Semiring multiplications on commutative monoids

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1. Introduction

Let $(H, +)$ denote a commutative monoid with identity 0. A *semiring multiplication* defined on $(H, +)$ is a binary operation $*: H \times H \to H$ such that

1. $(H, *)$ is a semigroup,

2. $a*(b+c) = a*b+a*c$ and $(b+c)*a = b*a+c*a$ for $a, b, c \in H$.

If it is also required that

3. $0*a=a*0=0$ for $a \in H$,

then $(H, +, *)$ is called a *hemiring*. For convenience $a*b$ is denoted by ab. (The basic ideas and terminology for hemirings can be found in [3].)

In Section 2, the set $\mathfrak{M}$ of hemiring multiplications is constructed for an arbitrary commutative monoid $(H, +)$. These multiplications are characterized in terms of mappings from $(H, +)$ into $\operatorname{End}(H, +)$, the set of endomorphisms on $(H, +)$. In Section 3 the discussion is narrowed to the class of monogenic hemirings (which includes division hemirings) by confining consideration to multiplications determined by automorphisms on $(H, +)$. Structural decompositions of the multiplicative semigroup of monogenic hemiring are given.

The methods developed in Section 2 can be used with the help of a digital computer to find all the hemirings on semigroups of low order. Such a computational project is in progress and will be the subject of a future paper.

In the sequel $|X|$ denotes the cardinality of the set X. The endomorphism which takes every element into zero is denoted by θ and the identity endomorphism on H by 1.

2. Hemiring multiplications

The set $\operatorname{End}(H, +)$ of all endomorphisms on $(H, +)$ is itself a semiring under the operations of pointwise addition and composition of functions, i.e.

$$[\alpha+\beta](x) = \alpha(x)+\beta(x) \text{ and } [\alpha\circ\beta](x) = \alpha(\beta(x)).$$

Theorem 2.1. *Let $(H, +, *)$ be a hemiring and for each $a \in H$ define $\varphi_a(x)=ax$, for each $x \in H$. Then*

1. $\varphi_a \in \operatorname{End}(H, +)$,

2. $\varphi_a(0)=0$ *and* $\varphi_0=\theta$,

3. $\varphi_{a+b} = \varphi_a+\varphi_b$,

4. $\varphi_{ab} = \varphi_a \circ \varphi_b$,

5. *as a consequence of the above the mapping* $\Phi: a \to \varphi_a$ *is a homomorphism from* $(H, +, *)$ *onto* $(\Phi(H), +, \circ)$ *and the latter system is a hemiring.*

The proof of the theorem is straightforward and calculational and will be omitted.

It is possible to reverse this process. Let $\mathscr{E}(H)=\{\alpha \in \operatorname{End}(H, +): \alpha(0)=0\}$. Then $(\mathscr{E}(H), +, \circ)$ is a hemiring. Note that any mapping $f: H \to \mathscr{E}(H)$ defines a multiplication $*_f$ on $(H, +)$, via $a *_f b=f(a)(b)$. This operation is left distributive over "+". The following theorem gives sufficient conditions for $(H, +, *_f)$ to be a hemiring. (In view of Theorem 2.1, the conditions are also necessary.)

Theorem 2.2. *If* $f: H \to \mathscr{E}(H)$ *is a mapping satisfying*

1. $f(a+b) = f(a)+f(b)$,

2. $f(a *_f b) = f(a) \circ f(b)$,

3. $f(0)=\theta$,

then $(H, +, *_f)$ *is a hemiring.*

The proof is straightforward.

Note that f is a homomorphism from the hemiring $(H, +, *_f)$ onto the hemiring $(f(H), +, \circ)$. We call $(H, +, *_f)$ the *hemiring determined by* f.

Theorem 2.3. *Let* $(H, +, *_f)$ *and* $(H, +, *_g)$ *be the hemirings determined by* f *and* g *respectively. Then* $(H, +, *_f)$ *is isomorphic to* $(H, +, *_g)$ *if and only if there exists an automorphism* $\alpha \in \operatorname{Aut}(H, +)$, *such that for each* $x \in H$,

$$f(x) = \alpha^{-1} \circ g(\alpha(x)) \circ \alpha.$$

Proof. Let α be an isomorphism from $(H, +, *_f)$ onto $(H, +, *_g)$. Then for each $x, b \in H$, $\alpha(x *_f b)=\alpha(x) *_g \alpha(b)$ and $\alpha \in \operatorname{Aut}(H, +)$. (Note the latter yields $\alpha(0)=0$.) Therefore

$$\alpha(f(x)(b)) = g(\alpha(x))(\alpha(b)),$$

$$[\alpha \circ f(x)](b) = [g(\alpha(x)) \circ \alpha](b),$$

for each $b \in H$ and $\alpha \circ f(x) = g(\alpha(x)) \circ \alpha$ or $f(x) = \alpha^{-1} \circ g(\alpha(x)) \circ \alpha$. The steps are reversible, hence the converse holds as well.

Corollary 2.4. If f determines a hemiring multiplication $*_f$ on $(H, +)$ and $*_f$ determines a mapping Φ as in Theorem 2.1, then $f=\Phi$.

Let $\mathfrak{M}$ be the set of all hemiring multiplications on $(H, +)$ and let $\mathfrak{M}'$ be the set of all left distributive binary operations on $(H, +)$ which have the property that $0 * a=a * 0=0$. By Theorems 2.1 and 2.2 for $\mathfrak{M}$ and the obvious analogs for $\mathfrak{M}'$ each element of $\mathfrak{M}'$ is determined by some $f: H \to \mathscr{E}(H)$ such that $a * b=f(a)(b)$.

(Note that elements of $\mathfrak{M}'$ need not be associative operations nor necessarily right distributive; however, each $(H, +, *)$, where $*$ is in $\mathfrak{M}'$, is a seminear-ring [2].) Define

$$(*_f + *_g)(x, y) = *_f(x, y) + *_g(x, y).$$

Note that

$$(*_f + *_g)(x, y) = x *_f y + x *_g y = f(x)(y) + g(x)(y) =$$

$$= [f(x) + g(x)](y) = \big[[f+g](x)\big](y) = *_{f+g}(x, y).$$

Hence $*_f + *_g = *_{f+g}$. Thus $\mathfrak{M}$ and $\mathfrak{M}'$ are closed under "+".

Theorem 2.5. $(\mathfrak{M}, +)$ *and* $(\mathfrak{M}', +)$ *are commutative monoids.*

Let $\mathcal{T}$ be the set of all mappings from H into $\mathcal{E}(H)$ and $\mathcal{T}$ be the subset of $\mathcal{T}'$ described in the hypothesis of Theorem 2.2. Observe that $(\mathcal{T}, +)$ and $(\mathcal{T}', +)$ are commutative monoids, where "+" is pointwise addition of functions.

Theorem 2.6. *The mapping* $f \to *_f$ *is an isomorphism of* $(\mathcal{T}', +)$ *onto* $(\mathfrak{M}', +)$ *and (suitably restricted) of* $(\mathcal{T}, +)$ *onto* $(\mathfrak{M}, +)$.

Proof. Homomorphy is a consequence of $*_f + *_g = *_{f+g}$ and surjectivity follows from Theorem 2.1 and its analog for $\mathfrak{M}'$. If $*_f = *_g$, then for each $a, b \in H$, $*_f(a, b) = *_g(a, b)$ and $f(a)(b) = g(a)(b)$; this leads to $f = g$.

Corollary 2.7. $(\mathfrak{M}', +)$ is isomorphic to the direct product of $|H|$ copies of $(\mathcal{E}(H), +)$ and $(\mathfrak{M}, +)$ is isomorphic to a subsemigroup of this direct product.

Proof. $(\mathfrak{M}', +) \cong (\mathcal{T}', +) \cong \Pi(\mathcal{E}(H), +)$.

Corollary 2.8. If $|H| = n$, finite, and $|\mathcal{E}(H)| = m$, then $|\mathfrak{M}| \leqq m^n$.

The number of non-isomorphic hemirings defined on $(H, +)$ will be much lower than this upper bound m^n.

All hemirings on a given $(H, +)$ can be constructed using mappings from H into $\mathcal{E}(H)$. The ones so constructed can then be sorted into isomorphism classes using Theorem 2.3. Even for very small monoids this process requires vast numbers of calculations, checks, and sorting, too much for the unaided human. However, with the help of a high speed digital computer the process can be easily carried out for certain monoids of low order. The author is presently engaged in such a project. It is hoped that the examples generated will be instructive.

3. Monogenic hemirings

Let $(H, +, *_f)$ be a hemiring and define $A(H) = \{x \in H : xH = 0\}$, $P(H) = H - A(H)$, Aut $(H, +)$ to be the set of all automorphisms on $(H, +)$, and $I(H)$ to be the left identities of $(H, *_f)$.

Definition 3.1. A hemiring $(H, +, *_f)$ is *monogenic* if and only if $f(P(H))$ is a subgroup of Aut $(H, +)$.

Without ambiguity the sets $A(H)$, $I(H)$, and $P(H)$ can usually be denoted simply by A, I, and P respectively. Some immediate observations concerning these sets are collected in the next theorem.

Theorem 3.2. *Let $(H, +, *_f)$ be a monogenic hemiring. Then*

1. $f(x)=1$ *if and only if* $x \in I$; *hence* $I \neq \emptyset$,
2. $I=\{e \in H: e^2=e,\ e \neq 0\}$,
3. I, A, *and* P *are subsemigroups of* $(H, *_f)$,
4. $xH=H$ *and* $xP=P$ *for each* $x \in P$,
5. x *is left cancellable in* $(H, *_f)$ *if and only if* $x \in P$,
6. $(P, *)$ *is a right group* [1, p. 37],
7. A *is the kernel of the semiring homomorphism* f *and hence is a k-ideal*,
8. $PA=A$ *and* $P+A = P$,
9. *if* $f(x)$ *has order* n, *then* $x^n \in I$, $x^j \notin I$ *for* $0<j<n$, *and* $x^{n+1}=x$,
10. $(f(H), +, \circ)$ *is a division hemiring.*

Theorem 3.3. *If $(H, +, *_f)$ is a monogenic hemiring, then $P(H)$ is isomorphic to the direct product of the group $(f(P), \circ)$ and the semigroup $(I, *)$.*

Proof. *A left coset modulo I* is a complex of the form xI, $x \in P$. Two left cosets are identical or disjoint; so the set $\{xI: x \in P\}=P/I$ partitions P. The operation $aI \cdot bI = abI$ is well-defined on P/I and P/I is a monoid with identity I. The relation $xI \to f(x)$ from P/I onto $f(P)$ is an isomorphism; hence $(P/I, \cdot)$ is isomorphic to the group $(f(P), \circ)$. The relation $(xI, i) \to xi$, where (xI, i) is an element of the direct product $P/I \otimes I$, is an isomorphism onto $(P, *)$.

Theorem 3.4. *If $(H, +, *_f)$ is a monogenic hemiring and $(f(P), \circ)$ is periodic, then P is the disjoint union of $|I|$ groups each isomorphic to $(f(P), \circ)$.*

Proof. For each $i \in I$ define

$$P_i = \{x \in P: x^n=i, \text{ for some integer } n \geqq 1\}.$$

Using the fact that every element of $f(P)$ has finite order calculation shows that P_i is closed under $*$ and is a group (with identity i). The sets P_i, $i \in I$, are disjoint; so $\{P_i: i \in I\}$ is a partition of P and P' is the disjoint union of the $|I|$ sets P_i. Since f restricted to the semigroup $(P, *)$ has kernel I, the homomorphism obtained by restricting f to P_i has trivial kernel and hence is a monomorphism. For each $p \in P$ calculation reveals that $pi \in P_i$ and hence $f(pi)=f(p)$; so f restricted to P_i is an isomorphism onto $(f(P), \circ)$.

Corollary 3.5. If H is finite, then $|P| = |I| \cdot |f(P)|$ and $|H| = |A|+|I| \cdot |f(P)|$.

Corollary 3.6. If $(H, +, *_f)$ is a monogenic hemiring with no (non-zero) divisors of zero, then

1. $(H-\{0\}, *) \cong (f(H-\{0\}), \circ) \otimes (I, *)$,
2. $H-\{0\}$ is the disjoint union of $|I|$ groups each isomorphic to $(f(H-\{0\}), \circ)$,
3. $(H, +, ^*)$ is a division hemiring if and only if $|I|=1$, i.e., there is exactly one non-zero idempotent.

Division hemirings are monogenic, as are finite hemirings with the left cancellation property (i.e., $ax=ay$ and $a\neq 0$ implies $x=y$). A monogenic hemiring with no non-zero divisors of zero (a *hemidomain*) has the left cancellation property. The right cancellation property does not usually imply two-sided cancellation, even in the finite case; however, for monogenic hemirings it implies even more.

Theorem 3.7. *If* $(H, +, *_f)$ *is a monogenic hemiring satisfying the right cancellation property, then it is a division hemiring.*

PROOF. If $f(a)=f(b)$, then $ax=bx$ for some $x\neq 0$ and hence $a=b$; so f is an isomorphism onto the division hemiring $(f(H), +, \circ)$.

References

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