On the generators of cosine operator functions

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Dedicated to Professor A. Rapcsák on the occasion of his 60th birthday

A cosine operator function is a mapping C of the field of real numbers R into the linear topological space $B(X)$, the space of bounded linear operators in a Banach space X (real or complex), satisfying for $\xi, \eta \in R$

$$C(\xi+\eta)+C(\xi-\eta) = 2C(\xi)C(\eta), \quad C(0)=I$$

and continuous in the strong operator topology of $B(X)$ (cf. [2], [3]). The generator operator A of C can be defined e.g. as $Ax=C''(0)x\ (x \in D(A))$ with domain $D(A)=$ $=\{x \in X: C(\xi)x$ is twice continuously differentiable in $R\}$.

M. Sova has shown ([4], 4.9.) that if A is the generator of a cosine operator function in a real Banach space X, then A is the generator of a semigroup of class (C_0) in X. In [5] he gave an example showing essentially that the converse is not generally true.

Theorem *of Sova* (cf. [5], 4,13.). *If X is a real separable infinite-dimensional Hilbert space, then there exists an operator A such that A is the generator of a semigroup of class (C_0) but of no cosine operator function.*

Remark. The proof of this theorem is not quite correct in [5], though the example given there is. We give here the corrected proof.

Proof. Let $\{e_i;\ i=0, 1, 2, \dots\}$ be a fixed orthonormal basis in X, $x \in X$, and $x_i=\langle x, e_i\rangle$ $(i=0, 1, 2, \dots)$. Define

$$D(A) = \left\{x \in X: \sum_{k=0}^{\infty} k^2(x_{2k}^2+x_{2k+1}^2) < \infty\right\}$$

and not $\sum_{k=0}^{\infty} k(x_{2k}^2+x_{2k+1}^2) < \infty$ as in [5]. For $x \in D(A)$ put $Ax = \sum_{k=0}^{\infty} k(x_{2k+1}e_{2k}-x_{2k}e_{2k+1})$. Then we get as in [5] $\langle Ax, y\rangle = -\langle x, Ay\rangle$ for $x, y \in D(A)$, thus $D(A) \subset D(A^*)$. On the other hand, suppose $y \in D(A^*)$, then $|\langle Ax, y\rangle| \leqq c\|x\|$ with $c \geqq 0$ for $x \in D(A)$. Define

$$y_n^{(p)} = \sum_{k=0}^{p} (-y_{2k+1}e_{2k}+y_{2k}e_{2k+1}) \cdot k^{1-\frac{1}{2^n}} \quad \text{for} \quad p = 1, 2, \dots,\ n = 0, 1, 2, \dots,$$

then $y_n^{(p)} \in D(A)$. For a fixed p we show by induction that

$$\|y_n^{(p)}\| \leqq c^{1-\frac{1}{2^n}} \|y\|^{\frac{1}{2^n}}$$

and

$$(*) \qquad \sum_{k=0}^{p} k^{2-\frac{1}{2^n}} (y_{2k}^2 + y_{2k+1}^2) \leqq c^{2-\frac{1}{2^n}} \|y\|^{\frac{1}{2^n}} \qquad (n = 0, 1, 2, \ldots).$$

Indeed, for $n = 0$

$$\|y_0^{(p)}\| = \left\{ \sum_{k=0}^{p} (y_{2k}^2 + y_{2k+1}^2) \right\}^{1/2} \leqq \|y\|$$

and

$$\sum_{k=0}^{p} k (y_{2k}^2 + y_{2k+1}^2) = |\langle A y_0^{(p)}, y \rangle| \leqq c \|y_0^{(p)}\| \leqq c \|y\|.$$

Moreover, if $(*)$ is true for a fixed n, then

$$\|y_{n+1}^{(p)}\|^2 = \sum_{k=0}^{p} k^{2-\frac{1}{2^n}} (y_{2k}^2 + y_{2k+1}^2) \leqq c^{2-\frac{1}{2^n}} \cdot \|y\|^{\frac{1}{2^n}}$$

and thus

$$\|y_{n+1}^{(p)}\| \leqq c^{1-\frac{1}{2^{n+1}}} \|y\|^{\frac{1}{2^{n+1}}}.$$

From this we obtain

$$\sum_{k=0}^{p} k^{2-\frac{1}{2^{n+1}}} (y_{2k}^2 + y_{2k+1}^2) = |\langle A y_{n+1}^{(p)}, y \rangle| \leqq c^{2-\frac{1}{2^{n+1}}} \|y\|^{\frac{1}{2^{n+1}}}.$$

If we introduce

$$M(y, c) = \sup_{0 \leqq n < \infty} c^{2-\frac{1}{2^n}} \|y\|^{\frac{1}{2^n}} < \infty,$$

we get for $p = 1, 2, \ldots$ that

$$\sum_{k=0}^{p} k^2 (y_{2k}^2 + y_{2k+1}^2) \leqq M(y, c) < \infty,$$

consequently $y \in D(A)$ and thus $A^* = -A$, A is normal and, being $\langle Ax, x \rangle = 0$ for $x \in D(A)$, bounded above. Hence A generates a semigroup of class (C_0) of normal operators in X, while no cosine operator function as it is shown in [5].

More can be stated on a cosine generator A if the underlying Banach space X is complex. H. O. Fattorini remarked in [2] that A then generates a semigroup U holomorphic in the right half plane and of class (C_0) in $(0, \infty)$, for which

$$U(\xi)x = (\pi \cdot \xi)^{-\frac{1}{2}} \int_0^\infty e^{-\eta^2/4\xi} C(\eta) x \, d\eta, \quad (x \in X, \ \operatorname{Re} \xi > 0)$$

where C is the cosine function generated by A. G. Da Prato and E. Giusti have even indicated that A generates a holomorphic semigroup of class $H(-\pi/2, \pi/2)$. Now we prove the following

Theorem. *In a complex Banach space X a cosine generator A also generates a semigroup of class $H(-\pi/2, +\pi/2)$, but the converse is generally not true.*

Proof. Let A generate a cosine operator function C with $\|C(\xi)\| \leqq Me^{\omega|\xi|}$ $(\xi \in R, M>0, \omega \geqq 0)$, then $\|\mu R(\mu^2; A)\| \leqq \frac{M}{\operatorname{Re}\mu-\omega}$ for $\operatorname{Re}\mu>\omega$. If

$$\lambda = |\lambda| \cdot e^{i\theta} (-\pi < \theta < +\pi)$$

with $(\operatorname{Im}\lambda)^2 > -4\omega^2 \operatorname{Re}\lambda + 4\omega^4$, then $\operatorname{Re}\sqrt{\lambda} = \operatorname{Re}\{|\lambda|^{1/2} e^{i\theta/2}\} = |\lambda|^{1/2} \cos(\theta/2) > \omega$, consequently

$$\|\lambda R(\lambda; A\| \leqq \frac{M}{\cos\left(\frac{\theta}{2}\right) - \frac{\omega}{|\lambda|^{1/2}}}.$$

Let $0 < \varepsilon < \pi$ and find $\lambda_\varepsilon > 0$ such that

1. $\{\lambda : |\arg(\lambda-\lambda_\varepsilon)| = \pi-\varepsilon\} \subset \{\lambda : (\operatorname{Im}\lambda)^2 > -4\omega^2 \operatorname{Re}\lambda + 4\omega^4\}$,

2. for $|\arg(\lambda-\lambda_\varepsilon)| = \pi-\varepsilon$ we have $\frac{\omega}{|\lambda|^{1/2}} < \frac{1}{2}\cos\left(\frac{\pi}{2}-\frac{\varepsilon}{2}\right)$.

For $\lambda \in \{\lambda : |\arg(\lambda-\lambda_\varepsilon)| < \pi-\varepsilon\}$ we have with $\theta = \arg\lambda$ that $|\theta/2| < \frac{\pi}{2}-\frac{\varepsilon}{2}$, thus $\cos(\theta/2) > \cos\left(\frac{\pi}{2}-\frac{\varepsilon}{2}\right)$, consequently

$$\|\lambda R(\lambda; A)\| \leqq \frac{2M}{\cos\left(\frac{\pi}{2}-\frac{\varepsilon}{2}\right)} = M_\varepsilon.$$

Therefore we get

$$\|R(\lambda; A)\| \leqq \frac{M_\varepsilon}{|\lambda|} \leqq \frac{M_\varepsilon}{d_\varepsilon(\lambda)}$$

where $d_\varepsilon(\lambda)$ denotes the distance of λ from the sector

$$\{\lambda : \pi \geqq |\arg(\lambda-\lambda_\varepsilon)| \geqq \pi-\varepsilon\},$$

and [1] (Theorem 12.8.1.) yields the first part of the theorem.

To prove the second part we essentially use an example in [1] (19.6). Define $\Delta = \{\lambda : \operatorname{Re}\lambda < -|\operatorname{Im}\lambda|^{3/2}\}$ and let $h(z)$ be a fixed function holomorphic in $\{z : |z| < 1\}$ and mapping this circle conformally upon Δ. Let X be the Banach space of complex functions f holomorphic and bounded in $\{z : |z| < 1\}$ and having the property that for every $\varepsilon > 0$ there exists an $M = M(\varepsilon, f)$ such that $|f(z)| \leqq \varepsilon$ on the set $\{z : |h(z)| \geqq M\}$, where $\|f\|$ is defined by $\sup_{|z|<1} |f(z)|$. Put $[Af](z) = h(z) \cdot f(z)$ with domain $D(A) = \{f \in X : h(z) f(z) \in X\}$, then A is the generator of the semigroup $[T(Z)f](z) = e^{Z \cdot h(z)} \cdot f(z)$. The maximal domain of analyticity of $T(Z)$ is the sector in which the support function $F(Z)$ of the closure of Δ is finite, that is $\{-\pi/2<\arg Z<\pi/2\}$. In this sector we have $\|T(Z)\|\leqq\exp\{F(Z)\}$, thus T is of class $H(-\pi/2,\pi/2)$. From [1], Theorem 19.6.1. we get that $\sigma(A)$ is the closure of Δ. On the other hand, if A were a cosine generator, then for some $\omega\geqq 0$ we should have

$$\sigma(A)\subset\{\lambda:(\operatorname{Im}\lambda)^2\leqq -4\omega^2\operatorname{Re}\lambda+4\omega^4\},$$

which does not hold here. Thus the proof is complete.

Bibliography

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