Growth of geometric means of an entire function

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1. Introduction

Let $f(z)$ be an entire function of order ϱ. Let ϱ_1 and λ_1 respectively, be the exponent of convergence and lower exponent convergence of the zeros of $f(z)$; so that

$$\lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log n(r)}{\log r} = \frac{\varrho_1}{\lambda_1} \quad (0 \leqq \lambda_1 \leqq \varrho_1 \leqq \infty), \tag{1.1}$$

where $n(r)$ represents the number of zeros of $f(z)$ in the disc

$$D \equiv \{z: |z| \leqq r\}.$$

Further, let

$$N(r) = \int_{r_0}^{r} \frac{n(x)}{x}\,dx. \tag{1.2}$$

Define the following mean values of $f(z)$:

$$G(r) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\Theta})|\,d\Theta\right], \tag{1.3}$$

$$g(r) = \exp\left[\frac{1}{2\pi r}\int_0^r\int_0^{2\pi} \log|f(xe^{i\Theta})|\,x\,dx\,d\Theta\right].$$

Further, let

$$g_k(r) = \exp\left[\frac{k+1}{2\pi r^{k+1}}\int_0^r\int_0^{2\pi} \log|f(xe^{i\Theta})|\,x^k\,d\Theta\,dx\right], \quad 0 < k < \infty. \tag{1.4}$$

Clearly, if $T(r)$ denotes the Nevanlinna's characteristic function, then

$$G(r) \leqq \exp\{T(r)\}$$

$$g_k(r) \leqq \exp\left[\frac{k+1}{r^{k+1}}\int_0^r T(x)x^k\,dx\right],$$

and so the orders of $g_k(r)$ and $G(r)$ do not exceed the orders of $f(z)$.

We remark here that the result of SRIVASTAVA [9]

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log G(r)}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix} \tag{1.5}$$

and of KUMAR [3]

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log g_k(r)}{\log r} = \begin{matrix}\varrho\\ \lambda\end{matrix}, \tag{1.6}$$

can be negated as $G(r)$ and $g_k(r)$ are solely exprecible in terms of the zeros. We can consider an entire function with large $M(r)$ (maximum modulus) and small number of zeros (for instance $f(z)=\exp(z)$ and $f(z)=\exp(z^p)\cos\sqrt{z}$ where p is any positive integer).*

In view of Jensen's theorem on the zeros of $f(z)$, we note that

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log G(r)}{\log r} = \begin{matrix}\varrho_1\\ \lambda_1\end{matrix} \quad (0 \leqq \lambda_1 \leqq \varrho_1 \leqq \infty), \tag{1.7}$$

and

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log g_k(r)}{\log r} = \begin{matrix}\varrho_1\\ \lambda_1\end{matrix} \quad (0 \leqq \lambda_1 \leqq \varrho_1 \leqq \infty). \tag{1.8}$$

In this paper, our aim is to study the growths of $G(r)$ and $g_k(r)$. The results are given in form of theorems.

2.

Theorem 1. *Let $f(z)$ be an entire function of exponent convergence ϱ_1 and lower exponent convergence λ_1, then*

$$\liminf_{r\to\infty} \frac{\log G(r)}{n(r)} \leqq \frac{1}{\varrho_1} \leqq \frac{1}{\lambda_1} \leqq \limsup_{r\to\infty} \frac{\log G(r)}{n(r)}.$$

PROOF. We first prove the latter half involving λ_1, supposing that $\lambda_1>0$. If this is not true, there will be a positive number j such that for all sufficiently large r,

$$\frac{\log G(r)}{n(r)} < \left(\frac{1}{\lambda_1} - j\right). \tag{2.2}$$

By Jensen's theorem

$$\log G(r) = \int_{r_0}^{r} \frac{n(x)}{x} + O(1).$$

* This fact has not been pointed out in the reviews of Srivastava's paper [9] MR [28 # 2216] and KUMAR's paper [3] MR [33 # 4280].

Substituting for $\log G(r)$ in (2.2), we have

$$\frac{\int\limits_{r_0}^{r} \frac{n(x)}{x}\,dx}{n(r)} < \left(\frac{1}{\lambda_1}-j\right)+O(1) \quad (r \to \infty)$$

or

$$\frac{n(r)}{\int\limits_{r_0}^{r} \frac{n(x)}{n}\,dx} > \left(\frac{1}{\lambda_1}-j\right)^{-1}+O(1) \quad (r \to \infty). \tag{2.3}$$

Therefore, by the integration of (2.3)

$$\log N(r) > \left(\frac{1}{\lambda_1}-j\right)^{-1} \log r + O(\log r)$$

which in virtue of Lemma 1.4 [1], leads to the contradiction

$$\lambda_1 = \liminf_{r\to\infty} \frac{\log N(r)}{\log r} \geqq \left(\frac{1}{\lambda_1}-j\right)^{-1}.$$

Similarly we prove the rest part of the theorem.

Remark. Our theorem is not only more general than Jain's Theorem 1 [2, Chapter 1] but has a different proof from his as well as shorter and more widely applicable.

3.

Theorem 2. *For an entire function $f(z)$ of exponent convergence ϱ_1 and lower exponent convergence λ_1, we have*

$$\limsup_{r\to\infty} \frac{\log G(r)}{n(r)\log r} \leqq 1-\frac{\lambda_1}{\varrho_1}. \tag{3.1}$$

Proof. When $\lambda_1=0$ or $\varrho_1=\infty$ (i.e. $\varrho_1^{-1}=0$), it is obvious from Jensen's theorem. Hence we suppose that $\lambda_1>0$, $\varrho_1<\infty$ and deduce from Jensen's theorem

$$\frac{\log G(r)}{n(r)\log r} = O(1)+1-J(r)P(r), \quad r \to \infty \tag{3.2}$$

where

$$J(r) = \frac{\int\limits_{r_0}^{r} \log x\, dn(x)}{\int\limits_{r_0}^{r} \log n(x)\, dn(x)}.$$

and

$$P(r) = \frac{\int\limits_{r_0}^{r} \log n(x)\, dn(x)}{n(r)\log r} = \frac{n(r)\log n(r)-n(r)+\text{a constant.}}{n(r)\log r}$$

Now

$$\liminf_{r\to\infty} J(r) \geqq \liminf_{r\to\infty} \frac{\log r}{\log n(r)} = \frac{1}{\varrho_1} \tag{3.3}$$

and

$$\liminf_{r\to\infty} P(r) = \liminf_{r\to\infty} \frac{\log n(r)}{\log r} = \lambda_1, \tag{3.4}$$

(3.3) and (3.4) in conjuction with (3.2), prove the theorem.

4.

Here we prove more sharper inequalities than those of SRIVASTAVA [10]. In what follows we shall prove the following:

Theorem 3. *For an entire function of finite order $\varrho>0$, we have*

$$\limsup_{r\to\infty} \frac{\log G(r)}{r^{\varrho}\log r} \leqq \limsup_{r\to\infty} \frac{n(r)}{r^{\varrho}} \leqq e\varrho T - \varrho t,$$

$$\liminf_{r\to\infty} \frac{\log G(r)}{r^{\varrho}\log r} \leqq \liminf_{r\to\infty} \frac{n(r)}{r^{\varrho}} \leqq \varrho t \tag{4.2}$$

and

$$\liminf_{r\to\infty} \frac{\log G(r)}{r^{\varrho}\log r} \leqq \liminf_{r\to\infty} \frac{n(r)}{r^{\varrho}} \leqq \lambda_1 T. \tag{4.3}$$

where T and t are type and lower type ($t\neq 0$) of $f(z)$ respectively.

For the proof of the theorem we require the following lemmas.

Lemma 1. *For any entire function of finite non-zero order ϱ*

$$e\varrho T \geqq \Delta + \varrho t \tag{4.4}$$

where

$$\Delta = \limsup_{r\to\infty} \frac{n(r)}{r^{\varrho}}.$$

PROOF. We suppose $f(0)\neq 0$. By Jensen's theorem

$$\log M(r) \geqq O(1) + \int_{r_0}^{r} \frac{n(x)}{x}\,dx.$$

Let $\Delta'>0$ such that $\Delta-\Delta'=\varepsilon>0$. Suppose $\frac{n(r_1)}{r_1^{\varrho}}>\Delta'$ where $r_1=r_1(\Delta')$. Then for all $r > r_1$

$$\log M(r) > O(1) + \log M(r_1) + \Delta' r_1^{\varrho} \int_{r_1}^{r} \frac{dx}{x} = \tag{4.5}$$

$$= O(1) + \log M(r_1) + \Delta' r_1^{\varrho} \left[\log \frac{r}{r_1}\right].$$

Also, it is possible to choose r_1 such that $\frac{\log M(r_1)}{r_1^{\varrho}} > t'$ with $t - t' = \varepsilon'$. Therefore, from (4.5) for all $r > r_1$, we obtain

$$\frac{\log M(r)}{r^{\varrho}} > \left(\frac{r_1}{r}\right)^{\varrho} \left[t' + \Delta' \log \frac{r}{r_1}\right] + O(r^{\varrho}). \tag{4.6}$$

Now, by the usual method of calculus we maximset the first term of right hand side of (4.6). We find its maxima which is attained for ihat value of r which satisfies the relation

$$\left(\frac{r}{r_1}\right) = \exp\left[\frac{\Delta' - \varrho t'}{\varrho \Delta'}\right] \tag{4.7}$$

and that maximum value is

$$\left(\frac{\Delta'}{\varrho}\right) \exp\left[\frac{\varrho t' - \Delta'}{\Delta'}\right].$$

Therefore from (4.6), we get

$$\frac{\log M(r)}{r^{\varrho}} > \left(\frac{\Delta'}{\varrho}\right) \exp\left[\frac{\varrho t' - \Delta'}{\Delta'}\right] + O(r^{\varrho}) \tag{4.8}$$

for r satisfying (4.7). We see that

$$\varrho T \geqq \Delta' \exp\left[\left(\frac{\varrho t'}{\Delta'}\right) - 1\right]. \tag{4.9}$$

Now, since Δ' can be fixed arbitrary close to Δ and t' arbitrary close to t, we immediately deduce from (4.9) the following result

$$\varrho T \geqq \Delta \exp\left[\left(\frac{\varrho t}{\Delta}\right) - 1\right].$$

Since for every real x, $e^x \geqq 1 + x$, we finally get

$$e\varrho T \geqq \Delta + \varrho t$$

or

$$\limsup_{r \to \infty} \frac{n(r)}{r^{\varrho}} \leqq e\varrho T - \varrho T.$$

Lemma 2. [5].

$$\liminf_{r \to \infty} \frac{n(r)}{r^{\varrho}} \leqq \varrho t. \tag{4.10}$$

Lemma 3.

(4.11)
$$\liminf_{r\to\infty}\frac{n(r)}{r^{\varrho}} \leqq \lambda_1 T.$$

PROOF. It is known that

(4.12)
$$\liminf_{r\to\infty}\frac{n(r)}{\log M(r)} \leqq \lambda_1. \quad ([6])$$

Let

$$\frac{(nr)}{r^{\varrho}} = \frac{n(r)}{\log M(r)}\,\frac{\log M(r)}{r^{\varrho}}$$

It is well known that if $\varphi(x)$ and $\Phi(x)$ are two non-negative functions then

$$\liminf\{\varphi(x)\cdot\Phi(x)\} \leqq \liminf\varphi(x)\limsup\Phi(x).$$

Here $\frac{n(r)}{\log M(r)}$ and $\frac{\log M(r)}{r^{\varrho}}$ are non-negative, so we have

$$\liminf_{r\to\infty}\frac{n(r)}{r^{\varrho}} \leqq \liminf_{r\to\infty}\frac{n(r)}{\log M(r)}\limsup_{r\to\infty}\frac{\log M(r)}{r^{\varrho}}$$

which along with (4.12) and definition of type give us

$$\liminf_{r\to\infty}\frac{n(r)}{r^{\varrho}} \leqq \lambda_1 T.$$

PROOF OF THEOREM 3. From (1.3), we have

$$\log G(r) = O(1) + \int_{r_0}^{r}\frac{n(x)}{x}\,dx \leqq n(r)[\log r - \log r_0] + O(1)$$

or

$$\frac{\log G(r)}{r^{\varrho}\log r} \leqq \frac{n(r)}{r^{\varrho}} + O(r^{\varrho}).$$

Proceeding to limits and making use of Lemma 1, Lemma 2 and Lemma 3, we get the required inequalities.

5.

Inspite of the fact that the functions $\log G(r)$ and $\log g_k(r)$ have the same order and same lower order, it is to be noted that for an entire function $f(z)$ of exponent convergence ϱ_1 $(0<\varrho_1<\infty)$ the asymptotic relation

$$\log G(r) \sim \log g_k(r) \quad \text{as} \quad r\to\infty$$

need not be true always, as in the case of ordinary means [4]. Consider for instance an entire function

$$f(z) = \prod_{n=1}^{\infty}[1 + z/n^2]$$

for which

$$\log G(r) \sim r^{\frac{1}{2}}$$

and

$$\log g_k(r) \sim \frac{2(k+1)r^{\frac{1}{2}}}{(2k+3)}.$$

We give below a theorem which gives us information as to how the function $\log G(r)$ and $\log g_k(r)$ grow relative to each other as $r \to \infty$.

Theorem 4. *Let $f(z)$ be an entire function of finite lower exponent convergence λ_1, then*

$$(5.1)\qquad \liminf_{r\to\infty} \frac{\log G(r)}{\log g_k(r)} \leqq \left(\frac{\lambda_1}{k+1}\right)\left[\left(1+\frac{k+1}{\lambda_1}\right)^{\left(1+\frac{\lambda_1}{k+1}\right)}\right], \quad \lambda_1 > 0$$
$$\leqq 1, \qquad \lambda_1 = 0,$$

Proof. Since

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log G(r)}{\log r} = \frac{\varrho_1}{\lambda_1}, \quad (0 \leqq \lambda_1 \leqq \varrho_1 \leqq \infty).$$

Then, following Shah [7] there exists a lower proximate order $\lambda_1(r)$ $(0 \leqq \lambda_1 < \infty)$ relative to $\log G(r)$, satisfying the following conditions:

(i) $\lambda_1(r)$ is a non-negative continuous function of r for $r \geqq r_0 > 0$.

(ii) $\lambda_1(r)$ is differentiable for all $r > r_0$ except at isolated points at which $\lambda_1'(r-0)$ and $\lambda_1'(r+0)$ exist.

(iii) $\lim_{r\to\infty} r\lambda_1'(r)\log r = 0$

(iv) $\lim_{r\to\infty} \lambda_1(r) = \lambda_1$

(v) $r^{\lambda_1(r)} \leqq \log G(r)$ and $\liminf_{r\to\infty} \frac{\log G(r)}{r^{\lambda_1(r)}} = 1.$

From (i)—(iv) deduce that

$$(5.2)\qquad \lim_{r\to\infty} \frac{J(hr)}{J(r)} = h^{\lambda_1}, \quad h > 1 \quad \text{where} \quad J(r) = r^{\lambda_1(r)} \ \ ([8]).$$

From (1.4), we have

$$(5.3)\qquad \log g_k(r) = \frac{k+1}{r^{k+1}} \int_0^r \log G(x) x^k\, dx \leqq \log G(r).$$

Further

$$(5.4)\qquad \log g_k(R) = \frac{k+1}{R^{k+1}} \int_0^R x^k \log G(x)\, dx \geqq \frac{k+1}{R^{k+1}} \int_r^R x^k \log G(x)\, dx \geqq$$
$$\geqq \frac{R^{k+1}-x^{k+1}}{R^{k+1}} \log G(r).$$

Let $R = xr$, $x > 1$. Then

$$\log G(r) \leqq \frac{x^{k+1}}{x^{k+1}-1} \log g_k(xr)$$

and

$$1 = \liminf_{r\to\infty} \frac{\log G(r)}{J(r)} \leqq \frac{x^{k+1}}{x^{k+1}-1} \liminf_{r\to\infty} \frac{\log g_k(xr)}{J(r)}.$$

From which it follows that

$$\liminf_{r\to\infty} \frac{\log g_k(xr)}{J(r)} \geqq \frac{x^{k+1}-1}{x^{k+1}}. \tag{5.5}$$

Put

$$\frac{\log g_k(xr)}{J(r)} = \frac{\log g_k(xr)}{J(xr)} \frac{J(xr)}{J(r)}.$$

Here

$$\frac{\log g_k(xr)}{J(xr)} \quad \text{and} \quad \frac{J(xr)}{J(r)}$$

are non-negative and so that

$$\liminf_{r\to\infty} \frac{\log g_k(xr)}{J(r)} \leqq \liminf_{r\to\infty} \frac{\log g_k(r)}{J(r)} x^{\lambda_1}$$

by (5.2). This inequality with (5.5) gives us

$$\liminf_{r\to\infty} \frac{\log g_k(r)}{J(r)} \geqq \frac{x^{k+1}-1}{x^{k+\lambda_1+1}}.$$

Using this inequality and from the equality

$$\frac{\log G(r)}{\log g_k(r)} = \frac{\log G(r)}{J(r)} \frac{J(r)}{\log g_k(r)},$$

we get

$$\liminf_{r\to\infty} \frac{\log G(r)}{\log g_k(r)} \leqq \liminf_{r\to\infty} \frac{\log G(r)}{J(r)} \limsup_{r\to\infty} \frac{J(r)}{\log g_k(r)} \leqq \frac{x^{(k+\lambda_1+1)}}{x^{k+1}-1}. \tag{5.6}$$

Now, by the usual method of Calculus we minimize the right hand side of (5.6). We find that its minima is attained for that value of x which satisfies the relation

$$x = [(k+\lambda_1+1)/\lambda_1]^{1/(k+1)}, \quad \lambda_1 > 0.$$

Substituting this value of x in (5.6), we get

$$\liminf_{r\to\infty} \frac{\log G(r)}{\log g_k(r)} \leqq \frac{\lambda_1}{k+1} \left[1+\frac{k+1}{\lambda_1}\right]^{[1+\lambda_1/(k+1)]}.$$

In the case when $\lambda_1 = 0$, the minimum value of the right hand side of (5.6) is one as $x \to \infty$. This completes the proof of the theorem.

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