A few observations regarding continuous solutions of a system of functional equations

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The problem of the existence and uniqueness of the continuous solutions of the system of functional equations

$$(1)\qquad \varphi_i(x) = h_i(x; \varphi_1[f_1(x)], \dots, \varphi_1[f_n(x)]; \dots; \varphi_m[f_1(x)], \dots, \varphi_m[f_n(x)]),$$
$$i = 1, \dots, m,$$

in wich φ_i, $i=1, \dots, m$, are unknown functions, was investigated by J. KORDYLEWSKI in [2] in the case where f_k is the k-th iterate of a function f, $k=1, \dots, n$, under the hypothesis that the characteristic roots of a suitable matrix are less than one in absolute value. Here, we shall give a simpler condition which guarantees the existence and uniqueness of the continuous solutions of system (1) and we shall show that under suitable assumptions this unique solution fulfils a Lipschitz condition. Moreover, we shall prove a theorem about the continuous dependence on the given functions for continuous solutions of this system. The proofs will be based on J. MATKOWSKI's results given in [3].

1. Assume the following hypotheses:

(i) X is a topological space, whereas Y_i with the metric ϱ_i, $i=1, \dots, m$, are complete metric spaces;

(ii) $h_i: X\times Y_1^n\times\dots\times Y_m^n \to Y_i$, $i=1, \dots, m$, and $f_k: X\to X$, $k=1, \dots, n$, are continuous functions. Furthermore,

$$\varrho_i(h_i(x; y_{1,1}, \dots, y_{1,n}; \dots; y_{m,1}, \dots, y_{m,n}), h_i(x; \bar{y}_{1,1}, \dots, \bar{y}_{1,n}; \dots; \bar{y}_{m,1}, \dots, \bar{y}_{m,n})) \leqq$$
$$\leqq \sum_{j=1}^{m}\sum_{k=1}^{n} a_{i,j,k}\varrho_j(y_{j,k}, \bar{y}_{j,k}),$$

for every $x\in X$ and $y_{j,k}, \bar{y}_{j,k}\in Y_j$; $i,j=1, \dots, m$; $k=1, \dots, n$, where $a_{i,j,k}$; $i,j=1, \dots, m$; $k=1, \dots, n$, are positive constants.

Write

$$(2)\qquad b_{i,j} = \sum_{k=1}^{n} a_{i,j,k}, \qquad i,j = 1, \dots, m,$$

$$(3)\qquad b^1_{\lambda,\mu} = \begin{cases} b_{\lambda,\mu} & \text{for } \lambda\neq\mu \\ 1-b_{\lambda,\mu} & \text{for } \lambda=\mu \end{cases}; \qquad \lambda,\mu = 1, \dots, m,$$

$$b^{\varkappa+1}_{\lambda,\mu} = \begin{cases} b^{\varkappa}_{1,1} b^{\varkappa}_{\lambda+1,\mu+1} + b^{\varkappa}_{\lambda+1,1} b^{\varkappa}_{1,\mu+1} & \text{for} \quad \lambda \neq \mu \\ b^{\varkappa}_{1,1} b^{\varkappa}_{\lambda+1,\mu+1} - b^{\varkappa}_{\lambda+1,1} b^{\varkappa}_{1,\mu+1} & \text{for} \quad \lambda = \mu \end{cases}; \tag{4}$$

$$\varkappa = 1, \dots, m-1; \quad \lambda, \mu = 1, \dots, m-\varkappa.$$

Theorem 1. *Let hypotheses* (i) *and* (ii) *be fulfilled and suppose that X is a compact space. If*

$$0 < b^{\varkappa}_{\lambda,\lambda}, \qquad \varkappa = 1, \dots, m; \; \lambda = 1, \dots, m+1-\varkappa, \tag{5}$$

where the constants $b^{\varkappa}_{\lambda,\mu}$, $\varkappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\varkappa$, are defined by (2)—(4), *then system* (1) *has exactly one continuous solution $\varphi_i: X \to Y_i$, $i=1, \dots, m$. This solution is given by the formula*

$$\varphi_i(x) = \lim_{\nu \to \infty} \varphi_{i,\nu}(x), \qquad i = 1, \dots, m, \quad x \in X, \tag{6}$$

where

$$\varphi_{i,\nu+1}(x) = h_i(x; \varphi_{1,\nu}[f_1(x)], \dots, \varphi_{1,\nu}[f_n(x)]; \dots; \varphi_{m,\nu}[f_1(x)], \dots, \varphi_{m,\nu}[f_n(x)]), \tag{7}$$

$$i = 1, \dots, m, \quad \nu = 0, 1, 2, \dots, \quad x \in X,$$

and $\varphi_{i,0}$ is an arbitrary continuous map from X into Y_i, $i=1, \dots, m$.

Proof. Denote by $\mathscr{C}_i$ the complete metric space of all continuous functions $\varphi: X \to Y_i$ with the supremum metric d_i, $i=1, \dots, m$, and put

$$T_i(\varphi_1, \dots, \varphi_m)(x) = h_i(x; \varphi_1[f_1(x)], \dots, \varphi_1[f_n(x)]; \dots; \varphi_m[f_1(x)], \dots, \varphi_m[f_n(x)]), \tag{8}$$

$$\varphi_j \in \mathscr{C}_j; \quad i, j = 1, \dots, m, \quad x \in X.$$

By hypothesis (ii) we have that

$$T_i(\mathscr{C}_1 \times \dots \times \mathscr{C}_m) \subset \mathscr{C}_i, \qquad i = 1, \dots, m$$

and

$$d_i(T_i(\varphi_1, \dots, \varphi_m), T_i(\bar\varphi_1, \dots, \bar\varphi_m)) \leqq \sum_{j=1}^{m} b_{i,j} d_j(\varphi_j, \bar\varphi_j), \tag{9}$$

$$\varphi_j, \bar\varphi_j \in \mathscr{C}_j; \quad i, j = 1, \dots, m,$$

where $b_{i,j}$; $i, j=1, \dots, m$, are defined by (2). Thus we may apply Matkowski's theorem contained in [3] from which we obtain our assertion.

In the next theorem the compactness of X is replaced by the hypothesis

(iii) There exists a sequence $\{G_\tau\}$ of open sets such that $X = \bigcup \{G_\tau : \tau = 1, 2, \dots\}$, $G_\tau \subset G_{\tau+1}$ and $\bar{G}_\tau$ is compact, $\tau = 1, 2, \dots$. Moreover, $f_k(G_\tau) \subset G_\tau$ for $k=1, \dots, n$ and $\tau = 1, 2, \dots$.

Namely, we have

Theorem 2. *If hypotheses* (i)—(iii) *and condition* (5) *are fulfilled, where the constants $b^{\varkappa}_{\lambda,\mu}$, $\varkappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\varkappa$, are defined by* (2)—(4), *then system* (1) *has exactly one continuous solution $\varphi_i: X \to Y_i$, $i=1, \dots, m$. This solution is given by* (6) *and* (7), *where $\varphi_{i,0}$ is an arbitrary continuous map from X into Y_i, $i=1, \dots, m$.*

The proof of this theorem is similar to that given in [1], theorem 2.

2. Now, we shall give a theorem regarding a property of the solution just obtained. Suppose that

(iv) (X, ϱ) is a metric space and (Y_i, ϱ_i), $i=1, \dots, m$, are complete metric spaces;

(v) The functions $h_i: X\times Y_1^n\times\dots\times Y_m^n \to Y_i$, $i=1, \dots, m$, and $f_k: X\to X$, $k=1, \dots, n$, fulfil the conditions

$$\varrho_i(h_i(x; y_{1,1}, \dots, y_{1,n}; \dots; y_{m,1}, \dots, y_{m,n}), h_i(\bar{x}; \bar{y}_{1,1}, \dots, \bar{y}_{1,n}; \dots; \bar{y}_{m,1}, \dots, \bar{y}_{m,n})) \leqq$$

$$\leqq a_i \varrho(x, \bar{x}) + \sum_{j=1}^{m} \sum_{k=1}^{n} a_{i,j,k} \varrho_j(y_{j,k}, \bar{y}_{j,k}),$$

and

$$\varrho(f_k(x), f_k(\bar{x})) \leqq s_k \varrho(x, \bar{x}),$$

for all $x, \bar{x}\in X$ and $y_{j,k}, \bar{y}_{j,k}\in Y_j$; $i,j=1, \dots, m$; $k=1, \dots, n$, where a_i, $a_{i,j,k}$ and s_k; $i,j=1, \dots, m$; $k=1, \dots, n$, are positive constants.

Put

$$c_{i,j} = \sum_{k=1}^{n} a_{i,j,k} s_k, \qquad i,j = 1, \dots, m, \tag{10}$$

$$c^1_{\lambda,\mu} = \begin{cases} c_{\lambda,\mu} & \text{for } \lambda\neq\mu \\ 1-c_{\lambda,\mu} & \text{for } \lambda=\mu \end{cases}; \qquad \lambda, \mu = 1, \dots, m, \tag{11}$$

$$c^{\varkappa+1}_{\lambda,\mu} = \begin{cases} c^{\varkappa}_{1,1} c^{\varkappa}_{\lambda+1,\mu+1} + c^{\varkappa}_{\lambda+1,1} c^{\varkappa}_{1,\mu+1} & \text{for } \lambda\neq\mu \\ c^{\varkappa}_{1,1} c^{\varkappa}_{\lambda+1,\mu+1} - c^{\varkappa}_{\lambda+1,1} c^{\varkappa}_{1,\mu+1} & \text{for } \lambda=\mu \end{cases}; \tag{12}$$

$$\varkappa = 1, \dots, m-1; \quad \lambda, \mu = 1, \dots, m-\varkappa.$$

Theorem 3. *Let hypotheses* (iv) *and* (v) *be fulfilled and suppose that X is a compact space. If the constants $b^{\varkappa}_{\lambda,\mu}$ and $c^{\varkappa}_{\lambda,\mu}$, $\varkappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\varkappa$, defined by* (2)—(4) *and* (10)—(12) *fulfil conditions* (5) *and*

$$0 < c^{\varkappa}_{\lambda,\lambda}, \qquad \varkappa = 1, \dots, m; \quad \lambda = 1, \dots, m+1-\varkappa, \tag{13}$$

respectively, then system (1) *has exactly one continuous solution $\varphi_i: X\to Y_i$, $i=1, \dots, m$. This solution fulfils a Lipschitz condition.*

Proof. The first part of the above assertion evidently results from theorem 1. We have still to prove that the solution obtained fulfils a Lipschitz condition.

It follows from (11)—(13) that there exist positive numbers $l_1, \dots, l_m$ and a $\vartheta\in(0, 1)$, such that

$$\sum_{j=1}^{m} c_{i,j} l_j \leqq \vartheta l_i, \qquad i = 1, \dots, m$$

([3], Lemma). In view of the homogeneity of the above system we may assume that

$$\frac{a_i}{1-\vartheta} \leqq l_i, \qquad i = 1, \dots, m.$$

This means that the system

$$a_i+\sum_{j=1}^{m} c_{i,j}l_j \leqq l_i, \qquad i = 1, \dots, m, \tag{14}$$

has a positive solution l_i, $i=1, \dots, m$. Let $\mathscr{L}_i$ be the class of all functions $\varphi: X \to Y_i$ such that

$$\varrho_i(\varphi(x), \varphi(\bar{x})) \leqq l_i \varrho(x, \bar{x}), \quad x, \bar{x} \in X, \quad i = 1, \dots, m, \tag{15}$$

where l_i, $i=1, \dots, m$, are a positive solution of (14). We shall prove that the transformation T_i defined by (8) fulfils

$$T_i(\mathscr{L}_1 \times \dots \times \mathscr{L}_m) \subset \mathscr{L}_i, \qquad i = 1, \dots, m. \tag{16}$$

Indeed, suppose that $\varphi_i \in \mathscr{L}_i$, $i=1, \dots, m$, and $x, \bar{x} \in X$. Applying (8), hypothesis (v), (15), (10) and (14) we have

$$\varrho_i(T_i(\varphi_1, \dots, \varphi_m)(x), T_i(\varphi_1, \dots, \varphi_m)(\bar{x})) \leqq$$

$$\leqq a_i \varrho(x, \bar{x}) + \sum_{j=1}^{m} \sum_{k=1}^{n} a_{i,j,k} \varrho_j(\varphi_j[f_k(x)], \varphi_j[f_k(\bar{x})]) \leqq$$

$$\leqq \left(a_i + \sum_{j=1}^{m} c_{i,j} l_j\right) \varrho(x, \bar{x}) \leqq l_i \varrho(x, \bar{x}),$$

which shows that $T_i(\varphi_1, \dots, \varphi_m)$, $i=1, \dots, m$, fulfil condition (15), i.e. (16) holds. Moreover, condition (9) is fulfilled, where $\mathscr{C}_i$ and d_i, $i=1, \dots, m$, are defined as in the proof of theorem 1. By Matkowski's theorem the unique continuous solution $\varphi_i: X \to Y_i$, $i=1, \dots, m$, of system (1) must belong to $\mathscr{L}_i$, $i=1, \dots, m$, so it fulfils a Lipschitz condition.

Recalling once more the method of the proof of theorem 2 in [1] we obtain

Theorem 4. *If hypotheses* (iii)—(v) *and conditions* (5) *and* (13) *are fulfilled, where the constants* $b^{\varkappa}_{\lambda,\mu}$ *and* $c^{\varkappa}_{\lambda,\mu}$, $\varkappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\varkappa$, *are defined by* (2)—(4) *and* (10)—(12), *respectively, then system* (1) *has exactly one continuous solution* $\varphi_i: X \to Y_i$, $i=1, \dots, m$. *This solution fulfils a Lipschitz condition.*

3. In this section we shall give a theorem on the continuous depedence of continuous solution of the system (1) on the given functions. To this end consider a sequence of the systems of functional equations

$$\varphi_i(x) = h_{i,\nu}(x; \varphi_1[f_{1,\nu}(x)], \dots, \varphi_1[f_{n,\nu}(x)]; \dots; \varphi_m[f_{1,\nu}(x)], \dots, \varphi_m[f_{n,\nu}(x)]), \tag{17}$$

$$i = 1, \dots, m; \quad \nu = 0, 1, 2, \dots$$

and assume that

(vi) $h_{i,\nu}: X \times Y_1^n \times \dots \times Y_m^n \to Y_i$, $i = 1, \dots, m$, and $f_{k,\nu}: X \to X$, $k = 1, \dots, n$; $\nu = 0, 1, 2, \dots$, are continuous functions. Furthermore

$$\varrho_i(h_{i,\nu}(x; y_{1,1}, \dots, y_{1,n}; \dots; y_{m,1}, \dots, y_{m,n}), h_{i,\nu}(x; \bar{y}_{1,1}, \dots, \bar{y}_{1,n}; \dots; \bar{y}_{m,1}, \dots, \bar{y}_{m,n})) \leqq$$

$$\leqq \sum_{j=1}^{m} \sum_{k=1}^{n} a_{i,j,k} \varrho_j(y_{j,k}, \bar{y}_{j,k}),$$

for every $x \in X$ and $y_{j,k}, \bar{y}_{j,k} \in Y_j$; $i,j=1, \dots, m$; $k=1, \dots, n$; $\nu=0, 1, 2, \dots$, where $a_{i,j,k}$; $i,j=1, \dots, m$; $k=1, \dots, n$, are positive constants;

(vii) The sequences $\{h_{i,\nu}\}_{\nu=1}^{\infty}$ and $\{f_{k,\nu}\}_{\nu=1}^{\infty}$ tend uniformly on every compact set to $h_{i,0}$ and $f_{k,0}$, $i=1, \dots, m$; $k=1, \dots, n$, respectively.

In the proof of the theorem on the continuous dependence of the continuous solutions of system (1) we shall use the following

Lemma. *Let* (X_i, σ_i), $i=1, \dots, m$, *be a complete metric spaces and suppose that the transformations* $F_{i,\nu}: X_1 \times \dots \times X_m \to X_i$, $i=1, \dots, m$; $\nu=0, 1, 2, \dots$, *fulfil*

$$(18) \qquad \sigma_i\big(F_{i,\nu}(x_1, \dots, x_m),\ F_{i,\nu}(\bar{x}_1, \dots, \bar{x}_m)\big) \leqq \sum_{j=1}^{m} b_{i,j}\sigma_j(x_j, \bar{x}_j),$$

$$x_j, \bar{x}_j \in X_j; \quad i,j = 1, \dots, m; \quad \nu = 0, 1, 2, \dots,$$

with positive constants $b_{i,j}$; $i,j=1, \dots, m$, *and*

$$(19) \qquad F_{i,0}(x_1, \dots, x_m) = \lim_{\nu \to \infty} F_{i,\nu}(x_1, \dots, x_m), \quad x_j \in X_j; \quad i,j = 1, \dots, m.$$

If the constants $b^{\varkappa}_{\lambda,\mu}$, $\varkappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\varkappa$, *defined by* (3) *and* (4) *fulfil* (5), *then the system*

$$(20) \qquad x_i = F_{i,\nu}(x_1, \dots, x_m), \qquad i = 1, \dots, m; \quad \nu = 0, 1, 2, \dots,$$

has for every $\nu=0, 1, 2, \dots$ *exactly one solution* $x_{i,\nu} \in X_i$, $i=1, \dots, m$. *This solution is given by*

$$(21) \qquad x_{i,\nu} = \lim_{\tau \to \infty} x_{i,\nu,\tau}, \qquad i = 1, \dots, m; \quad \nu = 0, 1, 2, \dots$$

and

$$(22) \qquad x_{i,\nu,\tau+1} = F_{i,\nu}(x_{1,\nu,\tau}, \dots, x_{m,\nu,\tau}), \qquad i = 1, \dots, m; \quad \nu, \tau = 0, 1, 2, \dots,$$

where $x_{i,\nu,0}$ *is an arbitrary element of* X_i, $i=1, \dots, m$; $\nu=0, 1, 2, \dots$. *Moreover,*

$$(23) \qquad x_{i,0} = \lim_{\nu \to \infty} x_{i,\nu}, \qquad i = 1, \dots, m.$$

PROOF. The existence and uniqueness of solution $x_{i,\nu}$, $i=1, \dots, m$; $\nu=0, 1, 2, \dots$ of system (20) and formula (21) follows from Matkowski's theorem [3]. We shall show that (23) holds. Take $x_i \in X_i$, $i=1, \dots, m$, and put $x_{i,\nu,0}=x_i$ for every $i=1, \dots, m$ and $\nu=0, 1, 2, \dots$. Next, by (3)—(5), (22) and (19) we may choose a system of positive numbers $r_1, \dots, r_m$ and a $\vartheta \in (0, 1)$ such that

$$(24) \qquad \sum_{j=1}^{m} b_{i,j} r_j \leqq \vartheta r_i, \qquad i = 1, \dots, m,$$

and

$$\sigma_i(x_{i,\nu,1}, x_{i,\nu,0}) \leqq r_i, \qquad i = 1, \dots, m; \quad \nu = 0, 1, 2, \dots$$

(cf. [3], Lemma). By induction, applying (22), (18) and (24) we get

$$\sigma_i(x_{i,\nu,\tau+1}, x_{i,\nu,\tau}) \leqq \vartheta^{\tau} r_i, \qquad i = 1, \dots, m; \quad \nu, \tau = 0, 1, 2, \dots .$$

This fact, jointly with (21), shows that the sequence

$$\{\sigma_i(x_{i,\nu}, x_{i,0})\}_{\nu=1}^{\infty}, \qquad i = 1, \ldots, m,$$

is bounded. Let us put

$$(25) \qquad u_{i,\nu} = \sigma_i(x_{i,\nu}, x_{i,0}), \quad v_{i,\nu} = \sigma_i\big(F_{i,\nu}(x_{1,0}, \ldots, x_{m,0}), x_{i,0}\big), \quad i = 1, \ldots, m; \quad \nu = 1, 2, \ldots,$$

$$(26) \quad v_{i,\nu,1} = v_{i,\nu}, \quad v_{i,\nu,\tau+1} = \sum_{j=1}^{m} b_{i,j} v_{j,\nu,\tau} + v_{i,\nu}, \qquad i = 1, \ldots, m; \quad \nu, \tau = 1, 2, \ldots.$$

Since the sequence $\{u_{i,\nu}\}_{\nu=1}^{\infty}$, $i=1, \ldots, m$, is bounded, we may require that the numbers r_i, $i=1, \ldots, m$, satisfying (24) fulfil also

$$(27) \qquad u_{i,\nu} \leqq r_i, \qquad i = 1, \ldots, m; \quad \nu = 1, 2, \ldots.$$

Taking into account (20), (18), (25), (19) and (26) we obtain

$$(28) \qquad u_{i,\nu} \leqq \sum_{j=1}^{m} b_{i,j} u_{j,\nu} + v_{i,\nu}, \qquad i = 1, \ldots, m; \quad \nu = 1, 2, \ldots$$

and

$$(29) \qquad \lim_{\nu \to \infty} v_{i,\nu,\tau} = 0, \qquad i = 1, \ldots, m; \quad \tau = 1, 2, \ldots.$$

Recalling (28), (27), (24), (26) and the induction principle we have

$$u_{i,\nu} \leqq \vartheta^{\tau} r_i + v_{i,\nu,\tau}, \qquad i = 1, \ldots, m; \quad \nu, \tau = 1, 2, \ldots,$$

thus by (29)

$$\lim_{\nu \to \infty} u_{i,\nu} = 0, \qquad i = 1, \ldots, m,$$

i.e. (23) holds.

Now, we shall prove a theorem on the continuous dependence of the continuous solutions of system (1).

Theorem 5. *Let hypotheses* (iv), (vi) *and* (vii) *be fulfilled and suppose that X is a compact space. If the constants $b_{\lambda,\mu}^{\varkappa}$, $\varkappa=1, \ldots, m$; $\lambda, \mu=1, \ldots, m+1-\varkappa$, defined by* (2)—(4) *fulfil condition* (5), *then system* (17) *has for every $\nu=0, 1, 2, \ldots$ exactly one continuous solution $\varphi_{i,\nu}: X \to Y_i$, $i=1, \ldots, m$, and the sequence $\{\varphi_{i,\nu}\}_{\nu=1}^{\infty}$ tends to $\varphi_{i,0}$, $i=1, \ldots, m$, uniformly in X.*

Proof. Let $(\mathscr{C}_i, d_i)$, $i=1, \ldots, m$, be defined as in the proof of theorem 1, and put

$$T_{i,\nu}(\varphi_1, \ldots, \varphi_m)(x) =$$
$$= h_{i,\nu}(x; \varphi_1[f_{1,\nu}(x)], \ldots, \varphi_1[f_{n,\nu}(x)]; \ldots; \varphi_m[f_{1,\nu}(x)], \ldots, \varphi_m[f_{n,\nu}(x)]),$$
$$\varphi_j \in \mathscr{C}_j; \quad i, j = 1, \ldots, m; \quad \nu = 0, 1, 2, \ldots; \quad x \in X.$$

It follows from hypothesis (vi) that

$$T_{i,\nu}(\mathscr{C}_1 \times \ldots \times \mathscr{C}_m) \subset \mathscr{C}_i, \qquad i = 1, \ldots, m; \quad \nu = 0, 1, 2, \ldots$$

and

$$d_i(T_{i,\nu}(\varphi_1, \ldots, \varphi_m), T_{i,\nu}(\bar{\varphi}_1, \ldots, \bar{\varphi}_m)) \leqq \sum_{j=1}^{m} b_{i,j} d_j(\varphi_j, \bar{\varphi}_j),$$

$$\varphi_j, \bar{\varphi}_j \in \mathscr{C}_j; \quad i, j = 1, \ldots, m; \quad \nu = 0, 1, 2, \ldots,$$

where $b_{i,j}$; $i, j=1, \ldots, m$, are defined by (2). Moreover,

$$d_i(T_{i,\nu}(\varphi_1, \ldots, \varphi_m), T_{i,0}(\varphi_1, \ldots, \varphi_m)) \leqq$$

$$\leqq \sup_{x\in X} \varrho_i(h_{i,\nu}(x; \varphi_1[f_{1,\nu}(x)], \ldots, \varphi_1[f_{n,\nu}(x)]; \ldots; \varphi_m[f_{1,\nu}(x)], \ldots, \varphi_m[f_{n,\nu}(x)]),$$

$$h_{i,\nu}(x; \varphi_1[f_{1,0}(x)], \ldots, \varphi_1[f_{n,0}(x)]; \ldots; \varphi_m[f_{1,0}(x)], \ldots, \varphi_m[f_{n,0}(x)])) +$$

$$+ \sup_{x\in X} \varrho_i(h_{i,\nu}(x; \varphi_1[f_{1,0}(x)], \ldots, \varphi_1[f_{n,0}(x)]; \ldots; \varphi_m[f_{1,0}(x)], \ldots, \varphi_m[f_{n,0}(x)]),$$

$$h_{i,0}(x; \varphi_1[f_{1,0}(x)], \ldots, \varphi_1[f_{n,0}(x)]; \ldots; \varphi_m[f_{1,0}(x)], \ldots, \varphi_m[f_{n,0}(x)]))$$

for every $\varphi_j \in \mathscr{C}_j$; $i, j=1, \ldots, m$; $\nu=1, 2, \ldots$, so in view of hypotheses (vi), (vii) and of the compactness of X

$$T_{i,0}(\varphi_1, \ldots, \varphi_m) = \lim_{\nu\to\infty} T_{i,\nu}(\varphi_1, \ldots, \varphi_m), \qquad \varphi_j \in \mathscr{C}_j; \quad i, j = 1, \ldots, m.$$

Taking into account these facts and applying the above lemma we obtain our assertion.

It turns out that instead of the compactness of X we may assume that

(viii) There exists a sequence $\{G_\tau\}$ of open sets such that $X = \bigcup \{G_\tau : \tau = 1, 2, \ldots\}$, $G_\tau \subset G_{\tau+1}$ and $\bar{G}_\tau$ is compact, $\tau = 1, 2, \ldots$. Moreover, $f_{k,\nu}(G_\tau) \subset G_\tau$ for $k=1, \ldots, n$; *every* $\nu = 0, 1, 2, \ldots$ and $\tau = 1, 2, \ldots$.

Theorem 6. *If hypotheses* (iv), (vi)—(viii) *and condition* (5) *are fulfilled, where the constants* $b^{\varkappa}_{\lambda,\mu}$, $\varkappa = 1, \ldots, m$; $\lambda, \mu = 1, \ldots, m+1-\varkappa$, *are defined by* (2)—(4), *then system* (17) *has for every* $\nu = 0, 1, 2, \ldots$ *exactly one continuous solution* $\varphi_{i,\nu}: X \to Y_i$, $i = 1, \ldots, m$, *and the sequence* $\{\varphi_{i,\nu}\}_{\nu=1}^{\infty}$ *tends to* $\varphi_{i,0}$, $i = 1, \ldots, m$, *uniformly on compact subset of* X.

This theorem results from theorems 2 and 5, since every compact subset of X is contained in a G_τ.

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