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Existence and uniqueness of measure valued solutions for Zakai equation

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Abstract. In this paper we discuss the question of existence and uniqueness of measure valued solutions for Zakai equation. In recent papers [Ref 1, Ref 2] this question has been discussed for Zakai equation under the assumption that the corresponding measure valued process is absolutely continuous with respect to the invariant measure of the associated linear stochastic differential equation. This holds if the measure induced by the initial condition is absolutely continuous with respect to the invariant measure as mentioned above. In this paper we remove this restriction.

$2 \equiv 3 \pmod 4$

1. Introduction

We give a brief introduction to the filtering problem leading to the Zakai equation. The process to be filtered is governed by a class of semilinear stochastic differential equations given by

$$\begin{aligned} &dx = Axdt + F(x)dt + \sqrt{Q}dW \\ &x(0) = x_0, \end{aligned} \tag{1.1}$$

in a separable Hilbert space H. The observation process is governed by a stochastic differential equation in a finite dimensional space say R^d given by

$$\begin{aligned} &dy = h(x, y)dt + \sigma_0(y)dw^0 \\ &y(0) = 0. \end{aligned} \tag{1.2}$$

Here x is the process (not observable) taking values in an infinite dimensional Hilbert space, and y is the observed process taking values in a finite dimensional space. Finite dimensionality of observation is most natural in all applications.

The general problem of filtering is to find the best unbiased mean square estimate of a functional of the process $x(t)$ given the history of the observed process y up to time $t, t \geq 0$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t \uparrow, P)$ denote a filtered probability space and $\mathcal{F}_t^y \equiv \sigma\{y(s), s \leq t\}$ denote the smallest sigma algebra generated by the observed process y up to time $t, t \geq 0$. Let $\phi : H \to R$ be any continuous bounded function. The filtering problem as stated above can then be restated as follows: Find an $\mathcal{F}_t^y$ measurable process $\{\eta(t), t \geq 0\}$, such that

$$E\{(\eta(t) - \phi(x(t))^2 | \mathcal{F}_t^y\} = \min \text{ for all } t \in I \equiv [0, T].$$

It is well known that the best filter is given by

$$\begin{aligned} \eta^0(t) &= E\{\phi(x(t)) | \mathcal{F}_t^y\} \\ &= \int_H \phi(\xi) Q_t^y(d\xi) \equiv Q_t^y(\phi), \end{aligned} \tag{1.3}$$

where

$$Q_t^y(\chi_\Gamma) = P\{x(t) \in \Gamma | \mathcal{F}_t^y\} \tag{1.4}$$

for $\Gamma \in \Sigma_H$ with Σ_H denoting the σ-algebra of Borel subsets of H. This suggests that we must find the conditional probability measure Q_t^y which is an $\mathcal{F}_t^y$ adapted (probability) measure valued stochastic process.

It is known [1,2,7] that Q_t^y satisfies the Kushner equation (in the weak sense)

$$\begin{aligned} dQ_t^y(\phi) &= Q_t^y(\mathcal{A}\phi)dt + \langle Q_t^y(\phi h) - Q_t^y(\phi) Q_t^y(h), dz(t) \rangle, \\ z(t) &= y(t) - \int_0^t Q_s^y(h) ds, Q_0^y(\phi) = \Pi_0(\phi) \end{aligned} \tag{1.5}$$

where $\mathcal{A}$ is the differential generator of the Markov process x related to the operators A, F and Q, Π_0 is the measure on H induced by the initial state x_0 and the process z, called the innovation process, is a standard Brownian motion in R^d. Clearly this system is a nonlinear stochastic PDE in an infinite dimensional space.

It was shown by Zakai [Ref 9] that this equation can be simplified to a linear stochastic evolution equation. In general, the solution ofthis equation is a measure valued process and is related to the solution of Kushner equation (1.5) as follows: for any bounded Borel measurable function ϕ on H

$$Q_t^y(\phi) \equiv \mu_t^y(\phi)/\mu_t^y(1), \ t \geq 0.$$

The measure valued process process μ_t^y is governed by the stochastic differential equation

$$
\begin{aligned}
d\mu_t(\phi) &= \mu_t(\mathcal{A}\phi)dt + \langle \mu_t(\phi h), \Gamma_0^{-1/2} dv\rangle, \\
\mu_0(\phi) &= \Pi_0(\phi), \phi \in D(\mathcal{A}), \Gamma_0 \equiv \sigma_0\sigma_0^*,
\end{aligned}
\tag{1.6}
$$

where v is a standard Brownian motion in R^d in a suitable probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{F}}_t \uparrow, \tilde{P})$ [see Ref 2]. This is the unnormalized measure valued process. Our main objective in this paper is to prove the existence and uniqueness of solution of this equation without assuming that $\Pi_0 \prec \mu^0$ or that it has a density.

Note that equations (1.5) and (1.6) are derived under the assumption that the system noise W is uncorrelated with the measurement noise w^0. A more general equation is given by

$$
\begin{aligned}
d\mu_t(\phi) &= \mu_t(\mathcal{A}\phi)dt + \langle \mu_t(\mathcal{G}(t)\phi), dv\rangle, \\
\mu_0(\phi) &= \nu_0(\phi), \phi \in D(\mathcal{A}),
\end{aligned}
\tag{1.7}
$$

where the measure $\nu_0(K) \equiv \tilde{P}\{x_0 \in K | \mathcal{F}_0^y\}, K \in \Sigma_H$ and $\mathcal{G}(t)$ is either a bounded or an unbounded operator valued $\tilde{\mathcal{F}}_t$ adapted random process. This is the case when either one or both the coefficients h and σ_0 of the measurement dynamics (1.2) are dependent on y. If, however, both are independent of y the operator $\mathcal{G}$ is deterministic. For simplicity of presentation only, we use this assumption throughout the the rest of the paper.

2. Markov semigroup

Since the process $\{x(t), t \geq 0,\}$ solving equation (1.1) is Markov it follows from Fokker-Planck equation that the corresponding conditional probability measure $P(t, x, G), x \in H, t \geq 0, G \in \Sigma_H$, defines a semigroup $P_t, t \geq 0$, given by

$$
(P_t\phi)(x) \equiv \int_H \phi(y) P(t, x, dy),
$$

on the space of bounded Borel measurable functions on H, denoted by $B(\Sigma_H)$. The space $B(\Sigma_H)$ with its natural norm topology given by

$$
\|f\|_0 \equiv \sup\{|f(x)|, x \in H\}
$$

is a Banach space. It is clear that the semigroup P_t, $t \geq 0$, is a contraction, however it is not strongly continuous on $B(\Sigma_H)$ in its natural norm topology. In a recent paper [Ref 4] DA-PRATO-ZABCZYK gave a construction of a C_0-semigroup $S(t), t \geq 0$, on the Hilbert space $L_2(H, \mu^0)$, which is an extension of the Markov semigroup $P_t, t \geq 0$. The measure μ^0 is a symmetric Gaussian measure on H. This is based on several assumptions as presented below:

(H1):

(a): A is the infinitesimal generator of a C_0-semigroup, $T(t), t \geq 0$ in H satisfying

$$\|T(t)\|_{\mathcal{L}(H)} \leq Me^{-\omega t},\ t \geq 0, \omega > 0,\ M \geq 1$$

(b): Q is a positive, symmetric, bounded operator in H so that the operator Q_t given by

$$Q_t x \equiv \int_0^t T(s)QT^*(s)x ds,\ x \in H,\ t \geq 0,$$

is nuclear for all $t \geq 0$ and $\text{Sup}_{t \geq 0} TrQ_t < \infty$.

(c): W is a cylindrical Wiener process with values in H with $CovW(1) = I$.

(H2): F is a bounded Lipschitz mapping from H to H.

(H3): For all $t \geq 0, \mathcal{I}mT(t) \subset \mathcal{I}m(Q_t^{1/2})$.

(H4): The operator valued function $\Gamma(t) \equiv (Q_t^{-1/2}T(t)), t \geq 0$, is Laplace transformable where $Q_t^{-1/2}$ is the pseudo inverse of $Q_t^{1/2}$.

Let $D\phi$ and $D^2\phi$ denote the first and second Fréchet derivatives of the function $\phi : H \to R^1$, whenever they exist as elements of H and $\mathcal{L}(H)$. Define the operator $\mathcal{A}_0$ and $\mathcal{A}$ by

$$\mathcal{A}_0\phi \equiv (1/2)Tr(QD^2\phi) + (x, A^*D\phi), x \in H$$

and

$$\begin{aligned} D(\mathcal{A}_0) \equiv &\{\phi \in C_b^2(H) : D^2\phi \in \mathcal{L}_1(H), \\ &\text{Sup}_{x \in H} \|D^2\phi\|_{\mathcal{L}_1(H)} < \infty, \\ &\exists \psi \in C_b^2(H) : \phi(x) = \psi(A^{-1}x), x \in H\}, \\ &\tilde{\mathcal{F}}\phi \equiv \langle F(.), D\phi(.)\rangle, \phi \in W^{1,2}(H, \mu^0) \text{ and} \\ &\mathcal{A} \equiv (\bar{\mathcal{A}}_0 + \tilde{\mathcal{F}}), D(\mathcal{A}) = D(\bar{\mathcal{A}}_0), \end{aligned}$$

where $\mathcal{L}_1(H)$ is the space of nuclear operators in H and $C_b^k(H)$ is the space of bounded k-times Fréchet differentiable functions on H and $W^{1,2}(H, \mu^0)$

is the Sobolev space determined by the completion of $C^1(H)$ with respect to the norm topology given by

$$\|\phi\|^2_{W^{1,2}(H)} \equiv \int_H (|\phi|^2 + \|Q^{1/2}D\phi\|^2)\mu^0(dx)$$

with μ^0 being the invariant measure as mentioned earlier. We consider the semigroup $S(t), t \geq 0$, corresponding to the Kolmogorov operator associated with the nonlinear stochastic evolution equation (1.1). The following result is due to Da Prato and Zabczyk [Ref 4] which played a significant role in the papers [Ref 1, Ref 2].

Theorem 2.1. *Suppose the assumptions (H1)–(H4) hold. Then*

(a): the linear version of (1.1), with $F = 0$, has a unique invariant Gaussian measure μ^0 on Σ_H.

(b): the operator $\mathcal{A}$ generates a C_0-semigroup of bounded linear operators, $S(t), t \geq 0$, in $L_2(H, \mu^0)$ and it is the extension of the original Markov transition operator, $P_t, t \geq 0$, from $B(\Sigma_H)$ to $L_2(H, \mu^0)$.

(c): Further $D(\mathcal{A}) \subset W^{1,2}(H, \mu^0)$ and for $t > 0, S(t)$ is a family of compact operators in $L_2(H, \mu^0)$.

Now we consider the Zakai equation (1.7).

Definition 2.2. An $\mathcal{F}^y_t$ adapted measure valued random process $\{\mu_t, t \in I\}$ is said to be a mild solution of the Zakai equation (1.7) if it satisfies the stochastic integral equation

$$\mu_t(\phi) = \nu_0(S(t)\phi) + \int_0^t \langle \mu_s(\mathcal{G}S(t-s)\phi), dv(s)\rangle, \tag{2.1}$$

where $S(t), t \geq 0$, is the semigroup as given by Theorem 2.1.

Formally, for any $\phi \in D(\mathcal{A})$, using Ito-differential and the C_0-property of the semigroup S equation (1.7) follows from equation (2.1). Writing equation (1.7) as the evolution equation

$$d\mu_t = \mathcal{A}^*\mu_t dt + \mathcal{G}^*\mu_t dv(t)$$

and using the variation of constants formula involving the adjoint semigroup $S^*(t), t \geq 0$, equation (2.1) follows from (1.7). This justifies the above definition.

As mentioned in the previous section, we wish to prove the existence and uniqueness of solution of this integral equation without the assumption on absolute continuity of ν_0 with respect to the invariant measure μ^0. For existence and uniqueness results based on the later assumption the reader is referred to [Ref. 1, Ref. 2]. Uniqueness result for finite dimensional case was also proved by Rozovskii [8] following a different method. The argument is that the natural space where one should look for solutions of such equations is a suitable space of (random) measures and not $L_2(H,\mu^0)$. This is precisely what we are interested in. This will require that we sacrifice the C_0-property and revert back to the restriction of S to the space of bounded scalar valued functions on H. However we continue to use the same notation for the restriction. In general our approach also admits the initial measure ν_0 to be a random variable taking values from the space of finitely additive measures. We shall use the theory of vector measures.

3. Vector measures

Let $\mathcal{X}$ denote a topological Hausdorff space and let $\mathcal{F}_{\mathcal{X}}$ denote a field of subsets of the set $\mathcal{X}$ and $\mathcal{Z}$ a Banach space.

Definition 3.1. A function $\mu : \mathcal{F}_{\mathcal{X}} \mapsto \mathcal{Z}$ is called a finitely additive vector measure if for each pair of disjoint sets $K_1, K_2 \in \mathcal{F}_{\mathcal{X}}, \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$ and it is said to be countably additive if for any disjoint sequence $\{K_i\} \in \mathcal{F}_{\mathcal{X}}$, for which $\bigcup K_i \in \mathcal{F}_{\mathcal{X}}$, $\mu(\bigcup K_i) = \sum_{i\geq 1} \mu(K_i)$ in the norm topology of $\mathcal{Z}$.

Definition 3.2. A vector measure $\mu : \mathcal{F}_{\mathcal{X}} \mapsto \mathcal{Z}$ is said to be strongly additive if for every sequence of pairwise disjoint members $\{K_j\} \in \mathcal{F}_{\mathcal{X}}$, the series $\sum_{j\geq 1} \mu(K_j)$ converges in the norm topology of $\mathcal{Z}$.

Definition 3.3. (a): A vector measure $\mu : \mathcal{F}_{\mathcal{X}} \mapsto \mathcal{Z}$ is said to be of bounded variation if for each $\Gamma \in \mathcal{F}_{\mathcal{X}}$, $|\mu|(\Gamma) < \infty$ where

$$|\mu|(\Gamma) \equiv \sup_{\pi} \sum_{K\in\pi} \|\mu(K)\|_{\mathcal{Z}}$$

with the supremum taken over all partitions π of the set Γ into a finite number of disjoint members of $\mathcal{F}_{\mathcal{X}}$. (b): The vector measure is said to be of bounded semivariation if for each $\Gamma \in \mathcal{F}_{\mathcal{X}}$,

$$\|\mu\|(\Gamma) \equiv \sup\{|z^*\mu|(\Gamma) : z^* \in \mathcal{Z}^*, \|z^*\| \leq 1\} < \infty,$$

where $\mathcal{Z}^*$ is the dual of $\mathcal{Z}$ and $|z^*\mu|(.)$ is the variation of the real valued measure $z^*\mu$. In particular, we write

$$\|\mu\|_b \equiv \|\mu\|(\mathcal{X}).$$

It is well known that any vector measure of bounded semivariation has bounded range [Diestel and Uhl, Jr. Ref. 5, Proposition I.1.11] and therefore they are called bounded vector measures.

Throughout the rest of the paper we consider $\mathcal{X}$ to be a regular topological space [6, Definition 1, p15]. Let $\mathcal{F}_{\mathcal{X}} \equiv \mathcal{A}_{\mathcal{X}}(\Sigma_{\mathcal{X}})$ denote an algebra (the Borel σ-algebra) of subsets of the set $\mathcal{X}$. Let $M_{rba}(\mathcal{F}_{\mathcal{X}}, \mathcal{Z})$ denote the vector space of bounded finitely additive regular $\mathcal{Z}$-valued vector measures. Recall that a vector measure μ is said to be regular if for every $z^* \in \mathcal{Z}^*$, $z^*\mu$ is a regular scalar valued set function in the sense that for every $E \in \mathcal{F}_{\mathcal{X}}$ and $\epsilon > 0$, there exist sets $F \in \mathcal{F}_{\mathcal{X}}$ with $\bar{F} \subset E$ and $G \in \mathcal{F}_{\mathcal{X}}$ such that $\text{Int}(G) \supset E$ and $|z^*\mu(C)| < \epsilon$ for all $C \in \mathcal{F}_{\mathcal{X}}, C \subseteq G \setminus F$. Define

$$\|\mu\|_0 \equiv \sup\{\|\mu(K)\|_{\mathcal{Z}}, K \in \mathcal{F}_{\mathcal{X}}\}.$$

It is well known [see Dunford and Schwartz, Ref 6, p160] that furnished with this topology, $M_{rba}(\mathcal{A}_{\mathcal{X}}, \mathcal{Z})$ and $M_{rba}(\Sigma_{\mathcal{X}}, \mathcal{Z})$ are Banach spaces. Let $M_{rca}(\Sigma_{\mathcal{X}}, \mathcal{Z})$ denote the vector space of regular countably additive bounded $\mathcal{Z}$-valued vector measures. Then $M_{rca} \subset M_{rba}$ is a closed linear manifold and hence it is also a Banach space.

In general, spaces of bounded vector measures are intimately related to spaces of bounded linear operators as are scalar valued measures related to continuous linear functionals, core of Riesz representation theorems [see Dunford and Schwartz, Ref. 6, Theorems IV.5.1, Corollary IV.5.3, Theorem IV.6.2, Theorem IV.6.3].

Let $B(\mathcal{A}_{\mathcal{X}})$ denote the completion in the sup topology of the space of real valued simple functions on $\mathcal{X}$ given by linear combinations of characteristic functions of sets $\{E \in \mathcal{A}_{\mathcal{X}}\}$ and let $\mathcal{L}(B(\mathcal{A}_{\mathcal{X}}), \mathcal{Z})$ denote the Banach space of bounded linear operators from $B(\mathcal{A}_{\mathcal{X}})$ to $\mathcal{Z}$. Similarly defined are the Banach spaces $\mathcal{L}(B(\Sigma_{\mathcal{X}}), \mathcal{Z})$ and $\mathcal{L}(C(\mathcal{X}), \mathcal{Z})$ where $B(\Sigma_{\mathcal{X}})(C(\mathcal{X}))$, endowed with sup norm, is the Banach space of real valued bounded measurable (bounded continuous) functions on $\mathcal{X}$.

The following theorem is a special case of [5, Theorem I.1.13].

Theorem 3.4. *There is one to one correspondence between the vector spaces $\mathcal{L}(B(\mathcal{F}_{\mathcal{X}}), \mathcal{Z})$ and $M_{rba}(\mathcal{F}_{\mathcal{X}}, \mathcal{Z})$ which is indicated by*

$$\mathcal{L}(B(\mathcal{F}_{\mathcal{X}}), \mathcal{Z}) \Longleftrightarrow M_{rba}(\mathcal{F}_{\mathcal{X}}, \mathcal{Z}).$$

Further this is also an isometry so that if $L \leftrightarrow \mu$ for $L \in \mathcal{L}(B(\mathcal{F}_{\mathcal{X}}), \mathcal{Z})$ and $\mu \in M_{rba}(\mathcal{F}_{\mathcal{X}}, \mathcal{Z})$ then $\|L\| = \|\mu\|(\mathcal{X}) \equiv \|\mu\|_b$.

The following result, which has independent interest, is a generalization of a result of DUNFORD and SCHWARTZ from scalar to vector case.

Theorem 3.5. *Let $\mathcal{X}$ be a normal topological space, Z a reflexive Banach space and $M_{rba}(\mathcal{X}, Z)$ the space of regular bounded vector measures defined on the field generated by closed sets of $\mathcal{X}$. Then there is one to one correspondence between the vector spaces $\mathcal{L}(C(\mathcal{X}), \mathcal{Z})$ and $M_{rba}(\mathcal{X}, \mathcal{Z})$ which is indicated by*

$$\mathcal{L}(C(\mathcal{X}), \mathcal{Z}) \Longleftrightarrow M_{rba}(\mathcal{X}, \mathcal{Z}).$$

Further this is also an isometry so that if $L \leftrightarrow \mu$ for $L \in \mathcal{L}(C(\mathcal{X}), \mathcal{Z})$ and $\mu \in M_{rba}(\mathcal{X}, \mathcal{Z})$ then $\|L\| = \|\mu\|_b$.

PROOF. Let $\mu \in M_{rba}(\mathcal{X}, Z)$ and define the operator L_μ by

$$L_\mu(f) \equiv \int_{\mathcal{X}} f(x)\mu(dx).$$

We show that L_μ is a bounded linear operator from $C(\mathcal{X})$ to $\mathcal{Z}$. For any $z^* \in \mathcal{Z}^*$, we have

$$\begin{aligned} |z^*(L_\mu(f))| &= \left| \int_{\mathcal{X}} f(x)(z^*\mu)(dx) \right| \\ &\le \|f\|_0 |(z^*\mu)|(\mathcal{X}) \\ &\le \|f\|_0 \|\mu\|(\mathcal{X}) \|z^*\| = \|f\|_0 \|\mu\|_b \|z^*\|. \end{aligned}$$

The last inequality follows from the fact that a bounded vector measure has finite semivariation. It is apparent now that L_μ is a bounded linear operator showing the correspondence $M_{rba}(\mathcal{X}, Z) \Rightarrow \mathcal{L}(C(\mathcal{X}), Z)$. It remains to prove the reverse inclusion. Let $L \in \mathcal{L}(C(\mathcal{X}), \mathcal{Z})$. Then clearly $(z^*, f) \to \ell(z^*, f) \equiv z^*(Lf)$ is a continuous bilinear form on $\mathcal{Z}^* \times C(\mathcal{X})$ and, in particular, for each fixed but arbitrary $z^* \in \mathcal{Z}^*$, $f \to \ell(z^*, f)$ is a continuous linear functional on $C(\mathcal{X})$. Since $\mathcal{X}$ is a normal topological space it follows from Theorem IV. 6.2 [Dunford and Schwartz, p262] that there exists a unique real valued measure $\nu_{z^*} \in M_{rba}(\mathcal{X}) \equiv M_{rba}(\mathcal{X}, R)$ such that

$$\ell(z^*, f) = \int_{\mathcal{X}} f(x)\nu_{z^*}(dx)$$

for all $f \in C(\mathcal{X})$. Hence there exists a unique $\lambda \in M_{rba}(\mathcal{X}, Z^{**})$, determined by L alone, such that $\nu_{z^*}(.) = \lambda(.)z^*$ for all $z^* \in \mathcal{Z}^*$. Since $\mathcal{Z}$ is reflexive, it follows from this that $\mathcal{L}(C(\mathcal{X}), \mathcal{Z}) \Rightarrow M_{rba}(\mathcal{X}, \mathcal{Z})$. The conclusion of the theorem now follows from this.

For the correlated case and continuous h we shall use a slightly generalized version of this result in Theorem 4.2 below.

Corollary 3.6. *If $\mathcal{Z}$ is a reflexive Banach space then the elements of both $M_{rba}(\mathcal{F}_{\mathcal{X}}, \mathcal{Z})$, and $M_{rba}(\mathcal{X}, \mathcal{Z})$ are strongly additive.*

Proof. It suffices to prove for the first one. Let $\mu \in M_{rba}(\mathcal{F}_{\mathcal{X}}, \mathcal{Z})$ and let $L_\mu \in \mathcal{L}(B(\mathcal{F}_{\mathcal{X}}), \mathcal{Z})$ denote the operator given by

$$L_\mu f \equiv \int_{\mathcal{X}} f(x)\mu(dx).$$

Since this is a bounded linear operator and $\mathcal{Z}$ is reflexive, it maps bounded sets into relatively weakly compact subsets of $\mathcal{Z}$. Hence it follows from [Diestel and Uhl. Jr. Ref. 5, Theorem VI.1.1, p 148] that μ is strongly additive.

We close this section with the remark that in view of the above results, in our treatment of the filtering problem, we look at vector measures as bounded linear operators with domain and range in suitable Banach spaces.

4. Zakai equations on the space of vector measures

Now we return to Zakai equation (1.7) or equivalently the corresponding integral equation (2.1). By a mild solution of equation (1.7) we always mean a solution of the integral equation (2.1) in the classical sense. As mentioned earlier, existence of solution of this equation has been been proved only for the case when the initial measure is deterministic and absolutely continuous with respect to the invariant measure μ^0. In this situation one has density valued solutions [see Refs. 1,2] in $L_2(H, \mu^0)$. We avoid this restrictive assumption and prove existence of solutions in the natural space of vector measures.

Consider the probability space $(\Omega, \mathcal{F}, \tilde{\mathcal{F}}_t \subset \mathcal{F}, \tilde{P})$. Define the Hilbert space $\mathcal{Z} \equiv L_2(\Omega, \tilde{P})$ with the norm topology $\|z\| \equiv (\tilde{E}|z|^2)^{1/2}$ and let Σ_H denote the σ-field of Borel subsets of the Hilbert space H. Let $M_{rba} \equiv M_{rba}(\Sigma_H, \mathcal{Z})$ denote the space of regular bounded finitely additive $\mathcal{Z}$-valued vector measures as introduced in section 2. We introduce the function space $B_\infty(I, M_{rba})$ to denote the linear space of functions defined on the time interval $I \equiv [0, T]$ taking values from the space M_{rba} and weakly measurable in the sense that for each $\phi \in B(\Sigma_H)$, $t \mapsto \mu_t(\phi) \equiv L_{\mu_t}(\phi)$ is

Lebesgue measurable and adapted to the σ-algebra $\tilde{\mathcal{F}}_t$, $t \in I$,and essentially bounded in norm. In other words, this is an equivalence class in the sense that for $\mu, \nu \in B_\infty(I, M_{rba})$, we identify μ with ν, ($\mu \approxeq \nu$), if

$$\tilde{P}\{\mu_t(\phi) = \nu_t(\phi), \text{ for all } \phi \in B(\Sigma_H), a.e.t \in I\} = 1.$$

Again, furnished with the norm topology,

$$\begin{aligned}\|\mu\|_{B_\infty(I,M_{rba})} &\equiv \text{ess}\,.\sup\{\|\mu_t\|_b, t \in I\}\\ &\equiv \text{ess}\,.\sup\{\|L_{\mu_t}\|_{\mathcal{L}(B(\Sigma_H),\mathcal{Z})}, t \in I\},\end{aligned}$$

it is a Banach space. By virtue of the isometric isomorphism as stated in Theorems 3.4, 3.5 and for the sake of economy of notations, we have written $\mu_t(\phi)$ for $L_{\mu_t}(\phi)$. Throughout the rest of the paper we shall use this convention.

4.1. Uncorrelated noise

We assume that h, arising in the observation equation, is a bounded Borel measurable map. In this case the operator $\mathcal{G}\in\mathcal{L}(B(\Sigma_H), B(\Sigma_H, R^d))$. This is the case where the dynamic noise is uncorrelated with the measurement noise.

Theorem 4.1. *Suppose $\mathcal{G} \in \mathcal{L}(B(\Sigma_H), B(\Sigma_H, R^d))$ and $S(t), t \geq 0$, the Markov semigroup as introduced in section 2. Then for each $\nu_0 \in M_{rba}$, equation (1.7) has a unique mild solution $\mu \in B_\infty(I, M_{rba})$. Further the solution is continuously dependent on the parameters $\{\nu_0, \mathcal{G}\}$.*

Proof. Let $\mathcal{K}$ denote the operator on $B_\infty(I, M_{rba})$ assuming values $(\mathcal{K}\mu)_t(\phi)$ at $t \in I, \phi \in B(\Sigma_H)$ given by

$$(\mathcal{K}\mu)_t(\phi) \equiv \nu_0(S(t)\phi) + \int_0^t \langle \mu_s(\mathcal{G}S(t-s)\phi), dv(s)\rangle.$$

First we show that $\mathcal{K}$ maps $B_\infty(I, M_{rba})$ into itself. Recall our notation $L_\nu(\phi) \equiv \nu(\phi)$. Since v is a standard Brownian motion on the probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{P})$, it is clear that

$$\tilde{E}((\mathcal{K}\mu)_t(\phi))^2 \leq 2\tilde{E}(\nu_0(S(t)\phi))^2 + 2\int_0^t \tilde{E}|(\mu_s(\mathcal{G}S(t-s)\phi))|^2_{R^d} ds,$$

for any $\phi \in B(\Sigma_H)$. The semigroup $S(t), t \in I$, restricted to $B(\Sigma_H)$, is a contraction and by assumption $\mathcal{G}$ is a bounded operator. Thus it follows from the isometric isomorphism due to Theorem 3.4, that there exists a constant $\gamma > 0$ such that

$$\tilde{E}((\mathcal{K}\mu)_t(\phi))^2 \le 2\biggl(\|\nu_0\|_b^2 + \gamma^2 \int_0^t \|\mu_s\|_b^2 ds\biggr)\|\phi\|_0^2,$$

for all $\phi \in B(\Sigma_H)$ and $t \in I$. Recalling our notation,

$$\tilde{E}((\mathcal{K}\mu)_t(\phi))^2 \equiv \|L_{(\mathcal{K}\mu)_t}(\phi)\|_{\mathcal{Z}}^2,$$

and again using Theorem 3.4, it follows from the preceding inequality that there exists a constant β dependent on γ and T such that

$$\sup\{\|(\mathcal{K}\mu)_t\|_b, t \in I\} \le \beta\biggl(\|\nu_0\|_b + \sup\{\|\mu_t\|_b, t \in I\}\biggr).$$

This shows that $\mathcal{K}$ maps $B_\infty(I, M_{rba})$ into $B_\infty(I, M_{rba})$. We prove that $\mathcal{K}$ has a fixed point in $B_\infty(I, M_{rba})$. It suffices to show that certain power of $\mathcal{K}$ is a contraction. For any pair of $\mu, \nu \in B_\infty(I, M_{rba})$, following similar computation, we have

$$\|(\mathcal{K}\mu)_t - (\mathcal{K}\nu)_t\|_b^2 \le \gamma^2 \int_0^t \|\mu_s - \nu_s\|_b^2 ds,$$

for $t \in I$. Hence by repeated substitution we obtain

$$\|(\mathcal{K}^n\mu)_t - (\mathcal{K}^n\nu)_t\|_b^2 \le \biggl(\gamma^{2n}/(n-1)!\biggr) \int_0^t (t-s)^{n-1}\|\mu_s - \nu_s\|_b^2 ds,$$

for $t \in I$.

Clearly it follows from this that

$$\|(\mathcal{K}^n\mu) - (\mathcal{K}^n\nu)\|_{B_\infty(I,M_{rba})} \le \alpha_n\|\mu - \nu\|_{B_\infty(I,M_{rba})},$$

where $\alpha_n \equiv \biggl((\gamma^2 T)^n/n!\biggr)^{1/2}$. Thus, for n sufficiently large, $\mathcal{K}^n$ is a contraction and hence both $\mathcal{K}^n$ and $\mathcal{K}$ has only one and the same fixed point. This proves that the integral equation (2.1) has a unique solution in $B_\infty(I, M_{rba})$ and hence equation (1.7) has a unique mild solution.

For continuity of solution with respect to the data, let μ^ν, μ^λ denote the unique solutions corresponding to the initial data $\nu, \lambda \in M_{rba}$ (respectively) for a fixed operator $\mathcal{G} \in \mathcal{L}(B(\Sigma_H), B(\Sigma_H, R^d))$. Similarly let $\mu^{\mathcal{G}}, \mu^{\mathcal{H}}$ denote the solutions corresponding to the operators $\mathcal{G}, \mathcal{H} \in \mathcal{L}(B(\Sigma_H), B(\Sigma_H, R^d))$ for a fixed initial state $\nu_0 \in M_{rba}$. Then, following similar steps, one can verify that there exist constants C_1 and C_2 such that

$$\|\mu_t^{\nu} - \mu_t^{\lambda}\|_b \leq C_1 \|\nu - \lambda\|_b, \quad t \in I,$$

$$\|\mu_t^{\mathcal{G}} - \mu_t^{\mathcal{H}}\|_b \leq C_2 \|\mathcal{G} - \mathcal{H}\|_{\mathcal{L}(B(\mathcal{A}_H), B(\mathcal{A}_H, R^d))}, \quad t \in I,$$

where the constants are dependent on the variables as indicated in their arguments

$$C_1 \equiv C_1(T, \|\mathcal{G}\|), C_2 \equiv C_2(T, \|\mathcal{G}\|, \|\mathcal{H}\|, \|\nu_0\|_b).$$

In other words the solution is globally Lipschitz with respect to the initial data (measure) and only locally Lipschitz with respect to the operator $\mathcal{G}$. This completes the proof.

4.2. Correlated noise

Here the system is driven by correlated noise in the sense that the measurement noise affects the system dynamics. The model is described as follows:

$$dx(t) = (Ax + F(x))dt + \sqrt{Q}dW_1 + B_1 dW_2,$$

$$x(0) = x_0 \in H$$

$$dy(t) = h(x)dt + B_2 dW_2,$$

$$y(0) = y_0.$$

The processes W_1 and W_2 are independent cylindrical Brownian motions in real separable Hilbert spaces H and R^d respectively. The operators $B_1 \in \mathcal{L}(R^d, H)$ and $B_2 \in \mathcal{L}(R^d, R^d)$ with B_2 being self adjoint and invertible. In this situation the Zakai equation is again given by

$$\begin{aligned} d\mu_t(\phi) &= \mu_t(\mathcal{A}\phi)dt + \langle \mu_t(\mathcal{G}\phi), dv \rangle \\ \mu_0 &= \nu_0, \phi \in D(\mathcal{A}), \end{aligned} \tag{4.1}$$

where the operator $\mathcal{G}$ is a first order differential operator (recall: D denotes the Fréchet differential) given by

$$\mathcal{G}\phi \equiv (B_2^{-1}h + B_1^* D)\phi, \phi \in D(\mathcal{A}),$$

which is clearly an unbounded operator on $B(\Sigma_H)$. Again, this can be written as the integral equation (2.1) with the difference being that, here $\mathcal{G}$ is an unbounded operator.

Under additional assumptions on the operator $\mathcal{A}$ and the initial measure, one can write this as a regular evolution equation in $L_2(H, \mu^0)$ and prove the existence and strong regularity properties of solutions [see Ref. 2, Theorem 4.12].

Here we were interested in measure valued solutions under only mild assumptions. Generalizing this line of approach to vector valued distributions it may be possible to prove a similar result for the correlated case. In this regard we believe that Theorem 3.5 will be useful with $\mathcal{X} = H$ and $C^k(H)$ in place of $C(H)$. We can only make the following comment.

Remark 4.2. We note that the solution of the Zakai equation for the correlated case can be approximated by vector measures as in Theorem 4.1. Replace the operator $\mathcal{G}$ of (4.1) by $\mathcal{G}_n \equiv n\mathcal{G}R(n, \mathcal{A})$ where $n \in \rho(\mathcal{A})$ with $R(n, \mathcal{A})$ denoting the resolvent of the operator $\mathcal{A}$. The sequence $\{\mathcal{G}_n\}$ is a family of bounded linear operators in $B(\Sigma_H)$ converging strongly to $\mathcal{G}$ on its domain $D(\mathcal{G}) \supset D(\mathcal{A})$. Hence by Theorem 4.1 there exists a sequence $\{\mu^n\} \in B_\infty(I, M_{rba})$ solving the problem (4.1) (in the mild sense) with $\mathcal{G}$ replaced by $\mathcal{G}_n$. We believe that this sequence converges to the solution of the problem (4.1) in the topology of $B_\infty(I, \mathcal{D}_{rba})$ where $\mathcal{D}_{rba}$ is a suitable space of Z-valued distributions on H. At this time it remains an open problem.

Remark 4.3. By virtue of Corollary 3.6, it follows from Theorems 4.1 that if μ is a mild solution of equation (1.7) then $\{\mu_t, t \in I\}$ is a family of strongly additive vector measures. However it is not obvious if the strong additivity holds uniformly in t on I. Another important question is: under what additional assumptions can we expect countable additivity?

Remark 4.4. Our results do not provide any information on the temporal regularity of solutions $t \to \mu_t$. For applications, however, it is desirable that functionals like

$$\Psi_g(\mu_t) \equiv \tilde{E}g\bigg(\langle\mu_t, \phi_1\rangle, < \mu_t, \phi_2\rangle, \ldots \langle\mu_t, \phi_n\rangle\bigg)$$

are continuous in $t \in I$ for any continuous scalar valued function g on R^n and any arbitrary family of functions $\phi_i \in B(\Sigma_H), 1 \leq i \leq n$ in case of Theorem 4.1.

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