On the definition of C^*-algebras

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Dedicadet to Professor András Rapcsák on his 60th Birthday

Let A be a linear associative algebra over the field C of the complex numbers. We do not assume that A has an identity element. A is called a normed algebra provided there is a linear space norm on A satisfying the so-called multiplicativity condition

$$(0) \qquad \|xy\| \leqq \|x\| \|y\| \quad (x, y \in A).$$

If A possesses an identity element 1, then it is also assumed that $\|1\|=1$. If A is complete with respect to this norm, then it is called a Banach algebra.

A mapping $x \to x^*$ of A into itself is called an involution if the following standard algebraic conditions

$$(i) \qquad (\lambda x+y)^* = \bar{\lambda} x^* + y^*$$

$$(ii) \qquad (xy)^* = y^* x^*$$

$$(iii) \qquad (x^*)^* = x \quad (\lambda \in C;\ x, y \in A)$$

are satisfied. An algebra with involution (*) is called an involutory algebra or briefly a *-algebra.

The starting point of the theory of Banach *-algebras was the excellent paper [6] of GELFAND and NAIMARK. In this work the authors proved that a Banach *-algebra with an identity is *-isomorphic and isometric to a norm-closed selfadjoint subalgebra of all bounded linear operators on a suitable Hilbert space provided the following there conditions are fulfilled:

$$(iv) \qquad \|x^* \cdot x\| = \|x^*\| \|x\|$$

$$(v) \qquad \|x^*\| = \|x\|$$

$$(vi) \qquad 1 + x^* x \quad \text{is regular} \quad (x \in A).$$

At the same time they conjectured that both the fifth and sixth conditions are consequences of the first four. As a consequence of the work by KELLEY—VAUGHT [10] and FUKAMIYA [5], KAPLANSKY was able to remowe in [9] the so-called symmetry condition (vi). In [8] GLIMM and KADISON have given a direct proof of the Gelfand—Naimark theorem without assuming conditions (v) and (vi). By a suitable embedding into the algebra obtained by adjunction of an identity to the original algebra, VOWDEN proved in [17] that no assumption on the existence of an identity is necessary to have the Gelfand—Naimark theorem.

By a B^*-algebra we shall mean a Banach *-algebra with condition (iv). It should be mentioned that for Banach *-algebras with unit Berkson [3] and Glickfeld [7] required the B^*-condition only with elements of the form $\exp(ih)$, h is selfadjoint. We have thus the characteristic identity

$$(\text{vii})_1 \qquad \|\exp(ith)\| = 1 \quad (t \text{ real}; \ h \in H(A)),$$

where $H(A)$ denotes the selfadjoint part of A.

This condition defines in a complex Banach algebra the set $H(A)$ of the so-called Hermitian elements. Palmer showed in [11, 12], that $(\text{vii})_1$ with the additional condition

$$(\text{vii})_2 \qquad H(A)+iH(A) = A$$

ensures the existence of an involution called Vidav-involution, such that $H(A)$ is just the selfadjoint part of the B^*-algebra A with respect to this involution.

In a Banach *-algebra A the conditions (iv), (v) are clearly equivalent to the so-called C^*-condition

$$(\text{viii}) \qquad \|x^*x\| = \|x\|^2 \quad (x \in A).$$

A Banach *-algebra with condition (viii) is called a C^*-algebra. In case A is a Banach *-algebra without a unit element, Behncke [2] and Elliott [4] proved that for A to be a B^*-algebra [or a C^*-algebra] it is enough that (iv) [or (viii)] hold when x is normal (i.e. such that $x^*x=xx^*$).

Recently H. ARAKI and G. A. ELLIOTT have shown in [1] that the multiplicativity condition (0) follows from the C^*-condition (viii) or from the B^*-condition (iv) provided the involution is a norm-continuous map. At the same time they raised the following two problems:

(I) Is it necessary to assume that the involution is continuous in the second statement?

(II) Is it enough to assume condition (iv) or (viii) for normal elements to conclude the multiplicativity condition (0)?

The answer to (II) is in the negative as the full algebra of bounded linear operators on a Hilbert space of dimension not less than 2 shows, with the numerical radius as norm [see e.g. 15]. However the author showed in [16], that the sub-C^*-condition

$$(\text{ix}) \qquad \|x^* \cdot x\| \leqq \|x\|^2 \quad (x \in A)$$

together with condition (viii) for normal elements ensures that a complex *-algebra A with complete linear space norm be a C^*-algebra.

The purpose of the present paper is to give an affirmative answer to the problem

(I) of the ARAKI—ELLIOTT paper [1]. More generally we shall prove that the sub-B^*-condition

$$\text{(X)} \qquad \|x^* \cdot x\| \leqq \|x^*\| \, \|x\| \quad (x \in A)$$

together with (iv) for normal elements implies the multiplicative condition (0) in complex *-algebras with complete linear space norm, i.e. A becomes a C^*-algebra. The last theorem completes the result of the author's paper [15].

We use in general the notation of RICKART's monograph [14]. The first statement plays a central role in the arguments of this paper. It is a general version of [15], Lemma 1.

Lemma 1. *Let A be a complex involutory algebra with linear space norm such that*

$$(1) \qquad \|x^* \cdot x\| \leqq C \|x^*\| \, \|x\| \quad (x \in A)$$

holds with some constant C independent on x.

Then A is a normed *-algebra with continuous involution under a suitable algebra-norm.

PROOF. We are going to prove first

$$(2) \qquad \|hk\| \leqq 4C \|h\| \, \|k\| \quad \big(h, k \in H(A)\big).$$

Let us consider the identity

$$4hk = (h+k)^2 - (h-k)^2 + i(h+ik) \cdot (h-ik) - i(h-ik) \cdot (h+ik)$$

for any two selfadjoint elements h and k. Apply then (1) together with the subadditive property of the linear space norm to have

$$\|hk\| \leqq C(\|h\| + \|k\|)^2.$$

In case that h and k differ from 0, we can replace h and k with $h/\|h\|$ and $k/\|k\|$ respectively so that (2) becomes immediate. Consider the "complexification norm" defined by

$$\|h+ik\|_1 = \frac{1}{\sqrt{2}} \sup \{\|h \cos t - k \sin t\| + \|h \sin t + k \cos t\| : t \text{ real}\}.$$

This norm is such that

$$\frac{1}{\sqrt{2}} (\|h\| + \|k\|) \leqq \|h+ik\|_1 \leqq \|h\| + \|k\|, \quad \|h\| = \|h\|_1$$

and

$$\|h+ik\|_1 = \|h-ik\|_1$$

hold [see 4, p. 7] for $h, k \in H(A)$. The multiplication is then continuous with respect to this norm as an easy computation shows:

$$\|xy\|_1 \leqq 8C \|x\|_1 \|y\|_1 \quad (x, y \in A).$$

The "extended left regular representation norm" on A with respect to 1-norm defines then an appropriate algebra-norm by

$$|x| = \sup \{\|x\lambda + xy\|_1 : \lambda \in C, y \in A; \ |\lambda| + \|y\|_1 = 1\} \quad (x \in A).$$

This norm $|.|$ satisfies indeed the multiplicative condition (0) since it is an operator norm on the normed space A, which is further equivalent to the 1-norm as the obvious inequality

$$\frac{1}{\sqrt{2}}\|x\| \leqq \|x\|_1 \leqq |x| \leqq 8C\|x\|_1 \quad (x\in A)$$

shows. This implies also that the involution is a continuous map on A under the $|.|$-norm so that the proof is complete.

The spectral radius on A with respect to the $|.|$-norm, defined by

$$r(x) = \lim_{n\to\infty} |x^n|^{1/n} \quad (x\in A), \tag{3}$$

will be of great importance in the following. It makes possible to solve the problem which remained open in the author's paper [15].

Theorem 2. *Let A be a complex commutative *-algebra with complete linear space norm such that*

$$\frac{1}{C}\|x^*\|\,\|x\| \leqq \|x^*x\| \leqq C\|x^*\|\,\|x\| \quad (x\in A) \tag{4}$$

with some constant C independent on x. Then A is a C^*-algebra with an equivalent norm.

Proof. We prove first that the spectral radius given by (3) satisfies the C^*-condition

$$r(x^*x) = r(x)^2 \quad (x\in A). \tag{5}$$

It is immediately an algebra pseudonorm such that

$$r(x^*) = r(x) \quad (x\in A) \tag{6}$$

being the *-operation continuous with respect to the $|.|$-norm.

Let now h be an arbitrary selfadjoint element in A. (4) gives then by induction for any natural n

$$C^{1-2^n}\|h\|^{2^n} \leqq \|h^{2^n}\| \leqq C^{2^n-1}\|h\|^{2^n}.$$

This implies the useful inequality for the spectral radius

$$\frac{1}{C}\|h\| \leqq r(h) \leqq C\|h\| \quad (h\in H(A)), \tag{7}$$

being the norms each equivalent on $H(A)$. It then follows

$$r(x) \leqq r\left(\frac{x+x^*}{2}\right) + r\left(\frac{x-x^*}{2i}\right) \leqq C(\|x\|+\|x^*\|) \quad (x\in A) \tag{8}$$

and

$$\|x\| \leqq \left\|\frac{x+x^*}{2}\right\| + \left\|\frac{x-x^*}{2i}\right\| \leqq C\big(r(x)+r(x^*)\big) = 2Cr(x) \quad (x\in A), \tag{9}$$

using only (6) for the canonical decomposition of x into selfadjoint components together with the subadditivity.

We are able to prove the following expression which is interesting in itself:

(10) $$r(x) = \lim_{n\to\infty} \max\,[\|x^n\|^{1/n}, \|(x^*)^n\|^{1/n}] \quad (x\in A).$$

Note first that for any natural n

$$r(x^n) = r(x)^n \quad (x\in A)$$

and so by (8)

$$r(x)^n = r(x^n) \leqq C(\|x^n\| + \|(x^*)^n\|) \leqq 2C \max(\|x^n\|, \|(x^*)^n\|)$$

holds with any $x\in A$. This shows

$$r(x) \leqq \liminf_{n\to\infty} \max(\|x^n\|^{1/n}, \|(x^*)^n\|^{1/n}) \quad (x\in A).$$

On the other hand we have by (9)

$$\max\,[\|x^n\|, \|(x^*)^n\|] \leqq 2Cr(x^n) = 2Cr(x)^n$$

giving the reverse inequality for $r(x)$ as follows:

$$\limsup_{n\to\infty} \max\,[\|x^n\|^{1/n}, \|(x^*)^n\|^{1/n}] \leqq r(x) \quad (x\in A).$$

These give together the stated identity (10).

We have thus in particular

(11) $$\lim_{n\to\infty} \min\,[\|x^n\|^{1/n}, \|(x^*)^n\|^{1/n}] = r(x^*x)/r(x) \quad (0\neq x\in A).$$

To show this let us consider by (4) the inequalities

$$\frac{1}{C}\min\,[\|x^n\|, \|(x^*)^n\|]\max\,[\|x^n\|, \|(x^*)^n\|] =$$

$$= \frac{1}{C}\|(x^*)^n\|\cdot\|x^n\| = \|(x^*x)^n\| \leqq C\|(x^*)^n\|\,\|x^n\| =$$

$$= C\min\,[\|x^n\|, \|(x^*)^n\|]\max\,[\|x^n\|, \|(x^*)^n\|];$$

take n^{th} rooth with any natural n.

We need only letting $n\to\infty$ to have (11) because of (10).

We are now going to prove the C^*-property (5) of the spectral radius. Assume first that there exists an index n_0 such that

(12) $$\|x^n\| = \min\,[\|x^n\|, \|(x^*)^n\|] \quad (n\geqq n_0)$$

holds for a nonzero x in A. We have then by (11)

$$\lim_{n\to\infty} \|x^n\|^{1/n} = r(x^*x)/r(x)$$

so that $\lambda^{-1}\cdot x$ has a quasi-inverse for $|\lambda| > r(x^*x)/r(x)$:

$$(\lambda^{-1}x)^0 = -\sum_{n=1}^{\infty} (\lambda^{-1}x)^n.$$

This implies the estimate for the spectral radius

$$r(x) \leqq r(x^*x)/r(x)$$

or

$$r(x)^2 \leqq r(x^*x).$$

The assumption

$$(12^*) \qquad \|(x^*)^n\| = \min[\|x^n\|, \|(x^*)^n\|] \quad (n \geqq n_0)$$

has by (6) the same consequence.

If neither of (12) and (12*) holds then we may assume without loss of the generality that

$$(13) \qquad 1 = r(x^*x)^{1/2} < r(x), \quad \|x\| \leqq \|x^*\|$$

and

$$\|(x^*)^n\| \leqq \|x^n\|, \quad \|x^{n+1}\| \leqq \|(x^*)^{n+1}\|$$

for infinitely n many integers. From the identity

$$4(x^*)^{n+1} = \sum_{k=0}^{3} i^k\big(x+(-i)^k(x^*)^n\big)(x^*+i^k x^n)$$

and using (4) together with the subadditive property of the norm, we get

$$\frac{1}{C}\|(x^*)^{n+1}\| \leqq \big(\|x\|+\|(x^*)^n\|\big)(\|x^*\|+\|x^n\|) \leqq$$

$$\leqq C\big(\|x^*x\|+\|(x^*x)^n\|\big)+\|(x^*)^n\|\cdot\|x^*\|+\|x\|\cdot\|x^n\|.$$

Noting that (4) also gives

$$\big(\min[\|x^n\|, \|(x^*)^n\|]\big)^2 \leqq \min[\|x^n\|, \|(x^*)^n\|]\max[\|x^n\|, \|(x^*)^n\|] =$$

$$= \|(x^*)^n\|\,\|x^n\| \leqq C\|(x^*x)^n\|,$$

we have from (13) by (7) and (9)

$$\frac{1}{C}\|(x^*)^{n+1}\| \leqq C^2\big(r(x^*x)+r(x^*x)^n\big)+Cr(x^*x)^{n/2}\cdot\|x^*\|+Cr(x^*x)^{1/2}\cdot\|x^n\| =$$

$$= C(2C+\|x^*\|)+C\|x^n\|.$$

Using the estimates

$$\|x^n\| \leqq 2Cr(x^n) = 2Cr(x)^n, \quad r(x)^{n+1} = r(x^{n+1}) \leqq C\big(\|x^{n+1}\| + \|(x^*)^{n+1}\|\big),$$

resulting from (8) and (9) we get by (4) and (12)

$$\frac{1}{C}r(x)^{n+1} - C = \frac{1}{C}r(x)^{n+1} - Cr(x^*x)^{n+1/2} \leqq \frac{1}{C}r(x)^{n+1} - C^{1/2}\cdot\|(x^*x)^{n+1}\|^{1/2} \leqq$$

$$\leqq \frac{1}{C}r(x)^{n+1} - \|x^{n+1}\| \leqq \|(x^*)^{n+1}\| \leqq C^2\cdot(2C + \|x^*\|) + 2C^3r(x)^n.$$

Letting $n\to\infty$ we have

$$\frac{1}{C}r(x)^{n+1} \leqq 2C^3r(x)^n$$

or

$$r(x) \leqq 2C^4.$$

Since in (13) $r(x^*x)=1$ was assumed, this implies for such an x in A for which (12) does not hold,

$$r(x) \leqq 2C^4r(x^*x)^{1/2}. \tag{14}$$

Because $C\geqq 1$, we have then (14) for any x in A. But an easy computation gives now

$$r(x)^n = r(x^n) \leqq 2C^4r\big((x^*)^n x^n\big)^{1/2} = 2C^4r(x^*x)^{n/2}, \qquad n = 1, 2, \dots$$

and taking a limit $n\to\infty$,

$$r(x)^2 \leqq r(x^*x) \quad (x\in A). \tag{5)$'$}$$

To have (5) we need only to prove the reverse to (5)$'$. But this is an easy consequence of (6) by the submultiplicative property of the spectral radius:

$$r(x^*x) \leqq r(x^*)r(x) = r(x)^2 \quad \text{for any} \quad x\in A.$$

We shall prove finally the equivalence of $r(x)$ to the original norm. Using (9), (5), (7) and (4) in this order we get for any x in A the inequalities:

$$\frac{1}{4C^2}\|x\|^2 \leqq r(x)^2 = r(x^*x) \leqq C\|x^*x\| \leqq C^2\|x^*\|\cdot\|x\|.$$

This implies the norm-continuity of the involution in the form

$$\|x^*\| \leqq 4C^4\|x\| \quad (x\in A)$$

and so

$$\frac{1}{2C}\|x\| \leqq r(x) \leqq 2C^3\|x\| \quad (x\in A). \tag{15}$$

Thus we get that A is a C^*-algebra with an equivalent norm given by (3), as asserted.

Remark. It should be remarked that the resulting continuity of the involution implies that the auxiliary norm $|.|$ is also equivalent to the original norm and that in place of (10) the improved expression

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} \quad (x\in A)$$

holds.

The main result of this paper is the following theorem. We get this result as a consequence of [15], Theorem 5 in view of the proof of Theorem 2, namely (5).

Theorem 3. *Let A be a complex involutory algebra with complete linear space norm such that*

$$\text{(X)} \qquad \|x^* x\| \leqq \|x^*\|\,\|x\| \quad \textit{for any} \quad x\in A$$

and

$$\text{(iv)}_{(n)} \qquad \|x^* x\| = \|x^*\|\,\|x\| \quad \textit{for any} \quad x^* x = xx^*$$

hold. Then A is a C^*-algebra with the original norm.

Proof. Let C be arbitrary maximal commutative selfadjoint subalgebra of A. It is not immediate that C is closed in A, being the continuity of the multiplication does not ensured. But the proof of Theorem 2 implies that $r(.)$ is an equivalent norm with the C^*-property (5) on C. In case if (12) holds, the quasi-inverse of $\lambda^{-1}x$ (for $|\lambda|>r(x^*x)/r(x)$) exists in A (not sure that at all in C) implying $r(x)^2=r(x^*x)$ also for such an x in C. In particular the multiplication in C is a continuous operation with respect to the original norm. The maximality of C now implies that C is closed in A, and so C is a C^*-algebra with an equivalent norm. As a consequence of [1], Theorem 2 C is then itself a C^*-algebra with the original norm.

We shall prove that A is a C^*-algebra also. Denote $A^\sim$ the $|.|$-norm completion of A. Then $A^\sim$ is a Banach *-algebra with respect to the norm $|.|$. To show that $A^\sim$ is a C^*-algebra with a norm equivalent to $|.|$, we need by [13], Corollary 12 only that the set

$$M = \left\{\sum_{n=1}^{\infty} (ih^\sim)^n/n! : h^{\sim *} = h^\sim \in A^\sim\right\}$$

is bounded in $(A^\sim, |.|)$. Since by (7) for normal $x\in A$ as an element of some maximal commutative selfadjoint subalgebra:

$$\frac{1}{\sqrt{2}}\|x\| \leqq |x| \leqq 8\left[\left\|\frac{x+x^*}{2}\right\| + \left\|\frac{x-x^*}{2i}\right\|\right] \leqq 16r(x) = 16\|x\|$$

holds (from Lemma 1), we have for a selfadjoint h in A

$$\frac{1}{\sqrt{2}}\left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\| \leqq \left|\sum_{n=1}^{\infty}(ih)^n/n!\right| \leqq 16\left\|\sum_{n=1}(ih)^n/n!\right\| \leqq 32$$

being the C^*-norm of the quasi-unitary element $\sum_{n=1}^{\infty}(ih)^n/n!$ is not greater than 2.

Let $h^\sim$ be now a selfadjoint element in $A^\sim$, then for any $\varepsilon > 0$ there exists h in $H(A)$ such that

$$|h| \leqq |h^\sim| \quad \text{and} \quad |h^\sim - h| < \varepsilon e^{-|h^\sim|}.$$

It then follows that

$$\left|\sum_{n=1}^{\infty} (ih^\sim)^n/n!\right| \leqq \left|\sum_{n=1}^{\infty} (ih)^n/n!\right| + \varepsilon \leqq 32 + \varepsilon$$

since

$$\left|\sum_{n=1}^{\infty} \frac{1}{n!}[(ih^\sim)^n - (ih)^n]\right| \leqq \sum_{n=1}^{\infty} \frac{1}{n!}\left|\sum_{m=0}^{n-1} (ih^\sim)^{n-m}(ih)^m - (ih^\sim)^{n-m-1}(ih)^{m+1}\right| =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}\left|\sum_{m=0}^{n-1} (ih^\sim)^{n-m-1}(ih^\sim - ih)(ih)^m\right| \leqq$$

$$\leqq |h^\sim - h| \cdot \sum_{n=1}^{\infty} \frac{|h^\sim|^{n-m-1}|h|^m}{(n-1)!} \leqq |h^\sim - h| e^{|h^\sim|} < \varepsilon.$$

Here ε was arbitrary so that 32 is a desired bound of M in norm $|.|$. The C^*-norm must agree for any x in A with $r(x^* x)^{1/2}$ as an easy consequence of the C^*-property. But (7) implies by the assumption

$$r(x^* x^*) = \|x^* \cdot x\| \quad (x \in A).$$

When we are able to prove

$$\|x\| \leqq \|x^* x\|^{1/2} \quad (x \in A), \tag{16}$$

then we have in fact

$$\|x\| = \|x^* x\|^{1/2} \quad (x \in A), \tag{17}$$

which proves the statement of the theorem. Indeed, since by assumption (X) we have

$$\|x\|^2 \leqq \|x^* x\| \leqq \|x^*\| \, \|x\|$$

for any x in A. This gives

$$\|x\| \leqq \|x^*\| \quad (x \in A)$$

or using (iii)

$$\|x^*\| = \|x\| \quad (x \in A).$$

Then (17) becomes clear from the estimates

$$\|x\| \leqq \|x^* x\|^{1/2} \leqq \|x^*\|^{1/2} \cdot \|x\|^{1/2} = \|x\| \quad (x \in A).$$

To prove (16) suppose that A possesses an identity element. Then [12], Corollary (3.7) ensures an appropriate expression of the C^*-norm for an x in A as follows:

$$\|x^* x\|^{1/2} = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : x = \right.$$

$$\left. = \sum_{j=1}^{n} \lambda_j \exp(ih_j^\sim), \quad h_j^{\sim *} = h_j^\sim \in A^\sim \ (j = 1, 2, \ldots), \ n = 1, \ldots\right\}.$$

If now $\sum_{j=1}^{n} |\lambda_j| < \|x^* x\|^{1/2} + \frac{\varepsilon}{2}$ for some $\varepsilon > 0$, chosing in A normal x_j $(j=1, 2, \ldots, n)$ such that

$$\|x_j\| = \|x_j^* x_j\|^{1/2} = 1, \quad |\exp(ih_j^{\sim}) - x_j| < \varepsilon/2\sqrt{2} \sum_{j=1}^{n} |\lambda_j| \qquad (j = 1, 2, \ldots, n),$$

we have

$$\|x\| \leqq \left\| x - \sum_{j=1}^{n} \lambda_j x_j \right\| + \left\| \sum_{j=1}^{n} \lambda_j x_j \right\| \leqq \sqrt{2} \sum_{j=1}^{n} |\lambda_j| |\exp(ih_j^{\sim}) - x_j| + \sum_{j=1}^{n} |\lambda_j| <$$

$$< \frac{\varepsilon}{2} + \|x^* x\|^{1/2} + \frac{\varepsilon}{2} = \|x^* x\|^{1/2} + \varepsilon$$

and so (16), being ε arbitrary.

In case if A has not an identity

$$\|x^* x\|^{1/2} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j| : x = \right.$$

$$\left. = \sum_{j=1}^{n} \lambda_j \sum_{m=1}^{\infty} (ih_j^{\sim})^m/m!, \quad h_j^{\sim *} = h_j^{\sim} \in A^{\sim} \;\; (j = 1, 2, \ldots, n), \; n = 1, 2, \ldots \right\}$$

holds similarly $\left(\text{where } \sum_{j=1}^{n} \lambda_j = 0\right)$. (16) can be shown in analogous way. The proof of theorem is thus complete.

We have the answer to the problem (I) as an immediate consequence of Theorem 3.

Corollary 4. Let A be a complex *-algebra with complete linear space norm with B^*-condition (iv).

Then A is a C^*-algebra.

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