Additive decomposition of the curvature tensor of a reducible tensorial connection

By NGUYEN V. TU (Debrecen)

To professor András Rapcsák on his 60th birthday

1. Introduction

An affine connection on a differentiable manifold M with local coefficients $\Gamma_j{}^i{}_k(x)$ induces a connection between the tensors. In the case of tensors of type $(2, 0)$ it defines an absolute differential. This is in local coordinates

$$Dt^{ij} \equiv dt^{ij} + (\Gamma_r{}^i{}_s \delta^j_k + \delta^i_r \Gamma_k{}^j{}_s) t^{rk} \, dx^s. \tag{1.1}$$

However (1.1) does not produce the most general possible linear mappings between the n^2 dimensional vector spaces formed by tensors of type $(2, 0)$ and attached to the different points of the base manifold. Such a general connection is given by

$$Dt^{ij} = dt^{ij} + \gamma_{rk}{}^{ij}{}_s(x) t^{rk} \, dx^s \tag{1.2}$$

and it is called a tensorial connection. Tensorial connections for arbitrary types of tensors were studied by a number of authors[1]). In the case of

$$\gamma_{rk}{}^{ij}{}_s = \Gamma_r{}^i{}_s \delta^j_k + \delta^i_r \Gamma_k{}^j{}_s$$

or

$$\gamma_{rk}{}^{ij}{}_s = \overset{(1)}{\Gamma}{}_r{}^i{}_s \delta^j_k + \delta^i_r \overset{(2)}{\Gamma}{}_k{}^j{}_s$$

the tensorial connection is reducible to Γ, resp. to the two different affine connections $\overset{(1)}{\Gamma}$ and $\overset{(2)}{\Gamma}$. Also we say in this case that γ is induced by Γ, resp. by $\overset{(1)}{\Gamma}$ and $\overset{(2)}{\Gamma}$.

(1.2) is linear in the components of the tensor. But also connections non-linear in the components of the tensor (resp. the vectors), e.g.

$$Dt^{ij} = dt^{ij} + \gamma^{ij}{}_s(x, t) \, dx^s,$$

were studied by several authors[2]).

---

[1]) A fairly extensive bibliography can be found for example in A. Cossu [1].

[2]) E.g. A. Kawaguchi [3].

We shall investigate non-linear connections of tensors of type $(m, 0)$ (tensorial connections) given by

$$Dt^\alpha = dt^\alpha + \gamma^\alpha{}_s(x, t)\, dx^s, \tag{1.3}$$

where $\alpha = i_1 i_2 \dots i_m$, and so $\alpha = 1, 2, \dots, N$; $N = n^m$. Connections (vectorial or tensorial) in this paper mean always non-linear connections (not necessarily linear connections) if not otherwise stated. We assume that: $\gamma^\alpha{}_s(x, t)$ is homogeneous of degree 1 in t^α and therefore (1.3) gets the form

$$Dt^\alpha = dt^\alpha + \gamma_\beta{}^\alpha{}_s(x, t)\, t^\beta\, dx^s \qquad (\beta = j_1 j_2 \dots j_m) \tag{1.4}$$

where

$$\gamma_\beta{}^\alpha{}_s(x, t) = \frac{\partial \gamma^\alpha{}_s(x, t)}{\partial t^\beta}.$$

Curvature tensors for these geometries can be obtained in two different ways. The first and more common way to obtain a curvature tensor is to express the conditions of integrability of the parallel displacement. This method furnishes a curvature tensor of order higher than 4, the order of the curvature tensor of a linear vector connection. The second method arives to a curvature tensor through investigation of the equivalence of two tensor connections. This way was taken by L. TAMÁSSY (e.g. in [5]). This provides a curvature tensor of order 4. In the case of a linear vector connection the two ways lead to the same curvature tensor (but not so for tensorial connections).

Also (1.4) can be reducible. Denote by

$$T_d = \{t^\alpha \mid t^{i_1 i_2 \dots i_m} = \xi^{i_1}_{(1)} \xi^{i_2}_{(2)} \dots \xi^{i_m}_{(m)}\} \subset T = \{t^\alpha\}$$

the set of those t^α which are products of m vectors $\xi^i_{(a)}$, $(a = 1, 2, \dots, m)$ and by

$$T_s = \{t^\alpha \mid t^{i_1 i_2 \dots i_m} = \xi^{i_1} \xi^{i_2} \dots \xi^{i_m}\} \subset T_d$$

the set of those t^α which are being generated by single vector ξ. Then $\gamma_\beta{}^\alpha{}_s$ is said to be reducible over T_d to the non-linear homogeneous vector connections with the coefficients $\overset{(a)}{H}{}_j{}^i{}_k(x, \xi_{(a)})$ if

$$\gamma_\beta{}^\alpha{}_s(x, t) = \sum_{a=1}^{m} \overset{(a)}{H}{}_{j_a}{}^{i_a}{}_s(x, \xi_{(a)})\, \delta^{i_1}_{j_1} \dots \delta^{i_{a-1}}_{j_{a-1}} \delta^{i_{a+1}}_{j_{a+1}} \dots \delta^{i_m}_{j_m} \quad (t \in T_d). \tag{1.5}$$

Similarly $\gamma_\beta{}^\alpha{}_s$ is reducible over T_s to a single vector connection $H_j{}^i{}_k(x, \xi)$ if

$$\gamma_\beta{}^\alpha{}_k(x, t) = \sum_{a=1}^{m} H_{j_a}{}^{i_a}{}_k(x, \xi)\, \delta^{i_1}_{j_1} \dots \delta^{i_{a-1}}_{j_{a-1}} \delta^{i_{a+1}}_{j_{a+1}} \dots \delta^{i_m}_{j_m} \quad (t \in Ts). \tag{1.6}$$

The last type curvature tensor is given by[3])

$$R_j{}^i{}_{kl} \equiv \frac{\partial M_j{}^i{}_k}{\partial x^l} - \frac{\partial M_j{}^i{}_l}{\partial x^k} + 2 M_s{}^i{}_{[k} M^s_{|j|l]}, \tag{1.7}$$

[3]) For $m = 2$ see L. TAMÁSSY [5].

where

$$(1.8) \qquad M_{i_1}{}^{j_m}{}_k \equiv \frac{1}{m} \gamma_{i_1}{}^{i_2 i_3 \dots i_m j_m}_{i_2 \dots i_m}{}_k.$$

In the case of a reducible tensorial connection, which case we will study, there exist the m curvature tensors

$$(1.9) \quad \overset{(a)}{R}{}_j{}^i{}_{kl}(x, \xi_{(a)}) \equiv \frac{\partial \overset{(a)}{H}{}_j{}^i{}_k(x, \xi_{(a)})}{\partial x^l} - \frac{\partial \overset{(a)}{H}{}_j{}^i{}_l(x, \xi_{(a)})}{\partial x^k} + 2\overset{(a)}{H}{}_s{}^i{}_{[k}(x, \xi_{(a)}) \overset{(a)}{H}{}^s_{|j|l]}(x, \xi_{(a)})$$

of the vector connections $\overset{(a)}{H}{}_j{}^i{}_k(x, \xi_{(a)})$ too.

In this paper we wish to study the relation between the curvature tensors (1.7) and (1.9), especially the case of the additive decomposability of R, i.e. the case when

$$(1.10) \qquad R_j{}^i{}_{kl}(x, t) = \frac{1}{m} \sum_{a=1}^{m} \overset{(a)}{R}{}_j{}^i{}_{kl}(x, \xi_{(a)}) \quad t \in T_d).$$

This problem was proposed by A. MOÓR.

We will find sufficient conditions for the additive decomposability of the curvature tensor R and we will study the vector connections to which the tensor connection reduces in the case of the decomposability of R.

The author wishes to offer to his teacher L. TAMÁSSY his thanks for his guidance during his stay at the *KOSSUTH University in Debrecen*.

2. Conditions for the decomposability of the tensor R

We will study reducible tensorial connections and therefore (1.5) or (1.6) and also $t \in T_d$ will be throughout assumed.

It follows from (1.7)—(1.10) by an easy computation that

$$(2.1) \qquad R_q{}^p{}_{kj}(x, t) = \frac{1}{m} \sum_{a=1}^{m} \overset{(a)}{R}{}_q{}^p{}_{kj}(x, \xi_{(a)}) + L_q{}^p{}_{kj}(x, t),$$

where

$$(2.2) \qquad L_q{}^p{}_{kj}(x, t) = \frac{2}{m^2} \Bigg\{ \sum_{\substack{a=1 \\ (a \neq b)}}^{m} \sum_{b=1}^{m} \overset{(a)}{H}{}_s{}^p{}_{[k}(x, \xi_{(a)}) \overset{(b)}{H}{}^s_{|q|j]}(x, \xi_{(b)}) -$$

$$- (m-1) \sum_{a=1}^{m} \overset{(a)}{H}{}_s{}^p{}_{[k}(x, \xi_{(a)}) \overset{(a)}{H}{}^s_{|q|j]}(x, \xi_{(a)}) \Bigg\}.$$

Since both R and $\overset{(a)}{R}$ are tensors, so is also L. It is obvious that the vanishing of L is equivalent with the additive decomposition of R.

Theorem 1. *If the tensorial connection* $\gamma_\beta{}^\alpha{}_s$ *is reducible over* $t\in T_d$ *to the vector connections* $\overset{(a)}{H}{}_j{}^i{}_k(x, \xi_{(a)})$ *with the property that in a coordinate system*

$$\overset{(a)}{H}{}_s{}^r{}_k(x, \xi_{(a)}) - \lambda_{kj}(x)\overset{(a)}{H}{}_s{}^r{}_j(x, \xi_{(a)}) = 0 \quad (*) \quad (a = 1, 2, \ldots, m) \tag{2.3}$$

then $R_j{}^i{}_{kl}(x, t)$ *is additively decomposable.*

$(*)$ means that summarising for indices occurring double in term will be omitted[4]).

From (2.3) we have

$$\overset{(a)}{H}{}_s{}^r{}_k(x, \xi_{(a)}) = \lambda_{kj}(x)\overset{(a)}{H}{}_s{}^r{}_j(x, \xi_{(a)}) = \lambda_{kj}(x)[\lambda_{jk}(x)\overset{(a)}{H}{}_s{}^r{}_k(x, \xi_{(a)})] \quad (*).$$

Putting $j=k$ in (2.3) we get

$$\overset{(a)}{H}{}_s{}^r{}_k(x, \xi_{(a)}) = \lambda_{kk}(x)\overset{(a)}{H}{}_s{}^r{}_k(x, \xi_{(a)}),$$

and thus

$$\lambda_{kj}(x)\lambda_{jk}(x) = \lambda_{kk}(x) = 1 \quad (*). \tag{2.4}$$

On the other hand (2.2) can be written in the form

$$L_q{}^p{}_{kj} \equiv \frac{2}{m^2}\sum_{a=1}^{m}\left\{\overset{(a)}{H}{}_s{}^p{}_k\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m}\overset{(b)}{H}{}_q{}^s{}_j\right) - (m-1)\overset{(a)}{H}{}_q{}^s{}_j\right] - \right.$$
$$\left. - \overset{(a)}{H}{}_s{}^p{}_j\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m}\overset{(b)}{H}{}_q{}^s{}_k\right) - (m-1)\overset{(a)}{H}{}_q{}^s{}_k\right]\right\}. \tag{2.5}$$

$L=0$ obviously if

$$\overset{(a)}{H}{}_s{}^p{}_k\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m}\overset{(b)}{H}{}_q{}^s{}_j\right) - (m-1)\overset{(a)}{H}{}_q{}^s{}_j\right] - \overset{(a)}{H}{}_s{}^p{}_j\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m}\overset{(b)}{H}{}_q{}^s{}_k\right) - (m-1)\overset{(a)}{H}{}_q{}^s{}_k\right] = 0. \tag{2.6}$$

Using (2.3) we get for the first term of the left hand side of (2.6)

$$\overset{(a)}{H}{}_s{}^p{}_k\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m}\overset{(b)}{H}{}_q{}^s{}_j\right) - (m-1)\overset{(a)}{H}{}_q{}^s{}_j\right] = \lambda_{kj}\lambda_{jk}\overset{(a)}{H}{}_s{}^p{}_j\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m}\overset{(b)}{H}{}_q{}^s{}_k\right) - (m-1)\overset{(a)}{H}{}_q{}^s{}_\kappa\right].$$

However, with respect to (2.4) (2.6) becomes true and since $L=0$ suffices for the additive decomposability of $'R_j{}^i{}_{kl}(x, t)$ the Theorem is proved.

---

[4]) For example in (2.3) we do not summarize for j.

We can see that (2.6) becomes true and thus the curvature tensor R of the reducible tensorial connection is also decomposable if

a) $$\overset{(a)}{H}{}_s{}^p{}_k(x,\xi) = \lambda_q^p(x,\xi)\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m} \overset{(b)}{H}{}_q{}^s{}_k(x,\xi) - (m-1)\overset{(a)}{H}{}_q{}^s{}_k(x,\xi)\right]\right] \quad (*)$$

or

b) $$\overset{(a)}{H}{}_s{}^p{}_k(x) = \lambda_q^p(x)\left[\left(\sum_{\substack{b=1\\(b\neq a)}}^{m} \overset{(b)}{H}{}_q{}^s{}_k(x) - (m-1)\overset{(a)}{H}{}_q{}^s{}_k(x)\right]\right]. \quad (*)$$

b) is independent from $\xi_{(a)}$ and therefore in this case R is decomposable for $t\in T_d$. a) can be considered as a linear equation system for the unknowns $\overset{(a)}{H}{}_q{}^s{}_k$ of number m for fixed s, q, k. This system has the detailed form:

(2.7)

$$\begin{cases} (m-1)\overset{(1)}{H}{}_q{}^s{}_k(x,\xi) - \overset{(2)}{H}{}_q{}^s{}_k(x,\xi) - \dots - \overset{(m)}{H}{}_q{}^s{}_k(x,\xi) = -\dfrac{1}{\lambda_q^p(x,\xi)}\overset{(1)}{H}{}_s{}^p{}_k(x,\xi), & (*) \\ -\overset{(1)}{H}{}_q{}^s{}_k(x,\xi) + (m-1)\overset{(2)}{H}{}_q{}^s{}_k(x,\xi) - \dots - \overset{(m)}{H}{}_q{}^s{}_k(x,\xi) = -\dfrac{1}{\lambda_q^p(x,\xi)}\overset{(2)}{H}{}_s{}^p{}_k(x,\xi), & (*) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ -\overset{(1)}{H}{}_q{}^s{}_k(x,\xi) - \overset{(2)}{H}{}_q{}^s{}_k(x,\xi) - \dots + (m-1)\overset{(m)}{H}{}_q{}^s{}_k(x,\xi) = -\dfrac{1}{\lambda_q^p(x,\xi)}\overset{(m)}{H}{}_s{}^p{}_k(x,\xi) & (*). \end{cases}$$

The sum of the left hand side vanishes and the sum of the right hand side is

(2.8) $$-\frac{1}{\lambda_q^p(x,\xi)}\sum_{a=1}^{m}\overset{(a)}{H}{}_s{}^p{}_k(x,\xi) \quad (*).$$

For (2.7) to be solvable (2.8) too has to vanish:

(2.9) $$\sum_{a=1}^{m}\overset{(a)}{H}{}_q{}^p{}_k(x,\xi) = 0$$

Using this in (2.7) we get

$$m\overset{(a)}{H}{}_q{}^s{}_k(x,\xi) = -\frac{1}{\lambda_q^p(x,\xi)}\overset{(a)}{H}{}_s{}^p{}_k(x,\xi) \quad (*) \qquad (a=1,2,\dots,m);$$

or by putting $\sigma_q^p = -m\lambda_q^p$

(2.10) $$\overset{(a)}{H}{}_s{}^p{}_k(x,\xi) = \sigma_q^p(x,\xi)\overset{(a)}{H}{}_q{}^s{}_k(x,\xi) \quad (*).$$

Conversely, from (2.9) and (2.10) we can easily get back the system a). The same holds for b) in the special case that $\overset{(a)}{H}{}_s{}^p{}_k$ is independent from ξ. Thus we have

Theorem 2. *If the tensorial connection is reducible over $t \in T_s$ (resp. $t \in T_d$) to the vector connections $\overset{(a)}{H}{}_j{}^i{}_k(x, \xi)$ (resp. $\overset{(a)}{H}{}_j{}^i{}_k(x)$) so that in a coordinate system*

a) $$\overset{(a)}{H}{}_s{}^r{}_k = \sigma_q^r H_q{}^s{}_k \quad (*)$$

and

b) $$\sum_{a=1}^{m} \overset{(a)}{H}{}_s{}^r{}_k = 0$$

hold, then $R_j{}^i{}_{kl}(x, t)$ is decomposable.

Theorem 3. a) *If the tensor connection $\gamma_\beta{}^\alpha{}_s(x, t)$ is reducible over $t \in T_s$ to the vector connection $H_j{}^i{}_k(x, \xi)$, then $R_j{}^i{}_{kl}(x, t)$ is decomposable for $t \in T_s$.*

b) *If $\gamma_\beta{}^\alpha{}_s$ is reducible over $t \in T_d$ to the linear vector connection $H_j{}^i{}_k(x)$, then $R_j{}^i{}_{kl}(x, t)$ is decomposable over $t \in T_d$.*

As a matter of fact, in case a) $\overset{(a)}{H}{}_j{}^i{}_k(x, \xi) = H_j{}^i{}_k(x, \xi)$ and so

$$L_q{}^p{}_{kj}(x, t) = \frac{2}{m^2}\{(m-1)mH_s{}^p{}_{[k}(x, \xi)H^s_{|q|j]}(x, \xi) - m(m-1)H_s{}^p{}_{[k}(x, \xi)H^s_{|q|j]}(x, \xi)\}$$

vanishes indentically. In case b) $\overset{(a)}{H}{}_j{}^i{}_k(x, \xi_{(a)}) = H_j{}^i{}_k(x)$, thus again $L \equiv 0$ and this suffices for the decomposability of R.

3. Existence of the vector connections of types (A) and (B)

We give methods to construct connections of the types

(A) $$H_j{}^i{}_k(x, \xi) = \lambda_{kl}(x) H_j{}^i{}_l(x, \xi) \quad (*)$$

and

(B) $$H_j{}^i{}_k(x, \xi) = \sigma_q^i(x, \xi) H_q{}^j{}_k(x, \xi) \quad (*)$$

which appeared in the previous paragraph.

I) *type* (A)

First we remark that the functions $\lambda_{kl}(x)$ have to satisfy the indentity

(3.1) $$\lambda_{kl}(x)\lambda_{lj}(x) = \lambda_{kj}(x) \quad (*).$$

This follows from (A) by repeated application of (A).

In order to construct a connection of type (A) consider n^2 functions $H_j{}^i{}_1(x, \xi)$ homogeneous of degree zero in ξ, and n never vanishing functions $f_i(x)$. Now let

$$\lambda_{kj}(x) = \frac{f_k(x)}{f_j(x)}$$

and

$$H_j{}^i{}_k(x, \xi) = \lambda_{k1}(x) H_j{}^i{}_1(x, \xi).$$

These λ_{kj} satisfy (3.1) and (2.4), and also (A) is satisfied, since

$$H_j{}^i{}_k(x,\xi)=\frac{\lambda_{k1}(x)}{\lambda_{l1}(x)}H_j{}^i{}_l(x,\xi)=\frac{f_k(x)}{f_1(x)}\frac{f_1(x)}{f_l(x)}H_j{}^i{}_l(x,\xi)=\lambda_{kl}(x)H_j{}^i{}_l(x,\xi)\quad(*).$$

II) *type* (B)

In the way similar to the previous considerations it can be seen that the σ^i_q-s satisfy the indentities

$$\sigma^i_i=1\quad(*)\quad\text{and}\quad\sigma^i_s\sigma^s_p=\sigma^i_p\quad(*).\tag{3.2}$$

In order to construct a connection of type (B) consider n never vanishing functions $H_1{}^j{}_1(x,\xi)$ homogeneous of degree zero in the ξ. Now let

$$H_1{}^j{}_2=\ldots=H_1{}^j{}_n=H_1{}^j{}_1,$$

$$\sigma^p_q(x,\xi)=\frac{H_1{}^p{}_k(x,\xi)}{H_1{}^q{}_k(x,\xi)},$$

and

$$H_j{}^i{}_k(x,\xi)=\sigma^i_1(x,\xi)H_1{}^j{}_k(x,\xi)\quad(*).$$

One can check that these functions satisfy (B) and (3.2).

Having constructed a connection of type (B) we can easily construct m connections satisfying conditions a) and b) of Theorem 2. If $H_j{}^i{}_k$ is of type (B), then $\overset{(1)}{H}=\overset{(2)}{H}=\ldots=\overset{(m-1)}{H}=H$ and $\overset{(m)}{H}=-(m-1)H$ obviously satisfy a) and b).

4. The non-invariance of the properties (A) and (B)

We show that neither of these properties is invariant with respect to transformations of the coordinates. We wish to show this for property (A). It suffices to show that there exist $H_j{}^i{}_k(x,\xi)$ and $x^i=x^i(x')$ such that the $H_j{}^i{}_k$ satisfy (A) but after the transformation $x^i=x^i(x')$ the transformed $H_{j'}{}^{i'}{}_{k'}$ do not satisfy (A) for any $\lambda_{k'l'}(x')$.

Let

$$H_j{}^i{}_k(x,\xi)\equiv f(x)\quad(\forall i,j,k).\tag{4.1}$$

These satisfy (A) with $\lambda_{kl}\equiv 1$.

Suppose that (A) holds after the transformation $x^i=x^i(x')$ for certain $\lambda_{k'l'}$. Then we have

$$H_{j'}{}^{i'}{}_{k'}=\lambda_{k'l'}H_{j'}{}^{i'}{}_{l'}\quad(*).\tag{4.2}$$

Taking into account the transformation law of the coefficients of the connection we have

$$\lambda_{k'l'}=\frac{A^{i'}_i A^j_{j'} A^k_{k'} H_j{}^i{}_k+(\partial_{k'}A^i_{j'})A^{i'}_i}{A^{i'}_i A^j_{j'} A^k_{l'} H_j{}^i{}_k+(\partial_{l'}A^i_{j'})A^{i'}_i}.$$

The right hand side must be independent from the indices i' and j', for so is the left hand side. Thus the right hand side gives the same value for two different pairs of indices i'_0, j'_0, i'_1, j'_1:

$$(4.3)\qquad \frac{A^{i'_0}_i A^j_{j'_0} A^k_{k'} H_j{}^i{}_k + (\partial_{k'} A^i_{j'_0}) A^{i'_0}_i}{A^{i'_0}_i A^j_{j'_0} A^k_{l'} H_j{}^i{}_k + (\partial_{l'} A^i_{j'_0}) A^{i'_0}_i} = \frac{A^{i'_1}_i A^j_{j'_1} A^k_{k'} H_j{}^i{}_k + (\partial_{k'} A^i_{j'_1}) A^{i'_1}_i}{A^{i'_1}_i A^j_{j'_1} A^k_{l'} H_j{}^i{}_k + (\partial_{l'} A^i_{j'_1}) A^{i'_1}_i}.$$

If (4.2) is correct, then we may replace $H_j{}^i{}_k$ in (4.3) by $f(x)$ .Then expressing from this the function $f(x)$ we have

$$(4.4)\qquad f(x) = F\big(A^i_{i'}, A^{i'}_i, \partial_{j'} A^i_{i'}, x'(x)\big)$$

with a relatively complicated, but well-defined function F; but since in (4.4) both $f(x)$ and the arguments of F are arbitrarily taken, (4.4) does not hold in general and so neither (4.2) can hold good for every transformation $x^i = x^i(x')$.

The statement concerning property (B) can be proved in a similar way.

5. A geometric property of a connection of type (A)

We wish to study the geometric character of the most simple, i.e. of a linear connection of type (A). Its coefficients satisfy

$$H_s{}^r{}_k(x) = \lambda_{\kappa j}(x) H_s{}^r{}_j(x) \quad (*).$$

We also assume that the $H_s{}^r{}_k$ are nonvanishing for any r, s, k.

In this connection

$$d\xi^i = -H_j{}^i{}_l(x)\xi^j\,dx^l = -\xi^j H_j{}^i{}_{k_0}(x)\lambda_{lk_0}(x)\,dx^l.$$

The question is under what conditions does $d\xi^i$ vanish for all ξ We remark that

$$\xi^j H_j{}^i{}_{k_0}(x) = \xi^1 H_1{}^i{}_{k_0}(x) + \xi^2 H_2{}^i{}_{k_0}(x) + \ldots + \xi^n H_n{}^i{}_{k_0}(x)$$

cannot be zero for every $\xi^1, \xi^2, \ldots, \xi^n$, for then $H_j{}^i{}_{k_0}$ would be zero for every i, j, k in contradiction to our assumption. Thus $d\xi^i$ vanishes for all ξ if and only if

$$(5.1)\qquad \lambda_{lk}(x)\,dx^l = 0.$$

This is a system of linear equations for dx^l. The solutions of this system are the directions in which the parallel displacement is the ordinary one.

In consequence of (3.1) the rows of the determinant $\Delta \equiv \mathrm{Det}\,|\lambda_{kl}|$ are proportional, and thus rank $\Delta = 1$. Therefore (5.1) is equivalent to the following equation:

$$(5.2)\qquad \lambda_{11}(x)\,dx^1 + \lambda_{12}(x)\,dx^2 + \ldots + \lambda_{1n}(x)\,dx^n = 0.$$

The directions dx satisfying (5.2) form a hyperplane at all points (x) and these hyperplanes establish an $(n-1)$ dimensional distribution $\Omega(x)$.

Let $\frac{\partial}{\partial x^i}(x)$ $(i=1, 2, \dots, n)$ be n local basic vectorfields spanning up a tangential space T_x of M at (x). These vectorfields are differentiable. Now we let us consider n vectorfields

$$(5.3)\qquad \begin{cases} X_\alpha(x) = \dfrac{\partial}{\partial x^\alpha}(x) - \dfrac{\lambda_{1\alpha}(x)}{\lambda_{1n}(x)}\dfrac{\partial}{\partial x^n}(x) & (\alpha = 1, 2, \dots, n-1). \\ X_n(x) = \dfrac{\partial}{\partial x^n}(x) \end{cases}$$

Their matrix is

$$(\tilde{\Delta}) = \begin{pmatrix} 1 & 0 & 0 \dots & -\dfrac{\lambda_{11}}{\lambda_{1n}} \\ 0 & 1 & 0 \dots & -\dfrac{\lambda_{12}}{\lambda_{1n}} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & -\dfrac{\lambda_{1(n-1)}}{\lambda_{1n}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $|\tilde{\Delta}|=1$, the vectors $X_i(x)$ $(i=1, 2, \dots, n)$ are linearly independent. Moreover, because the $\lambda_{ij}(x)$ are differentiable, so are the $X_i(x)$. We can see that $X_\alpha(x) \in \Omega(x)$. Therefore these $X_\alpha(x)$ can be regarded as a basic vectorfield of $\Omega(x)$. Thus $\Omega(x)$ is a differentiable distribution[5]).

It is known that a differentiable distribution is integrable if and only if it is involutive[6]). Now we want to find the conditions for the integrability of $\Omega(x)$. From (5.3) we have

$$(5.4)\qquad [X_\alpha, X_\beta] = \left[\frac{\partial(x)}{\partial x^\alpha} - \frac{\lambda_{1\alpha}(x)}{\lambda_{1n}(x)}\frac{\partial(x)}{\partial x^n}, \frac{\partial(x)}{\partial x^\beta} - \frac{\lambda_{1\beta}(x)}{\lambda_{1n}(x)}\frac{\partial(x)}{\partial x^n}\right] =$$

$$= \left\{\frac{\lambda_{1\alpha}(x)}{\lambda_{1n}(x)}\frac{\partial}{\partial x^n}\left(\frac{\lambda_{1\beta}(x)}{\lambda_{1n}(x)}\right) - \frac{\partial}{\partial x^\alpha}\left(\frac{\lambda_{1\beta}(x)}{\lambda_{1n}(x)}\right) + \frac{\partial}{\partial x^\beta}\left(\frac{\lambda_{1\alpha}(x)}{\lambda_{1n}(x)}\right) - \right.$$

$$\left. - \frac{\lambda_{1\beta}(x)}{\lambda_{1n}(x)}\frac{\partial}{\partial x^n}\left(\frac{\lambda_{1\alpha}(x)}{\lambda_{1n}(x)}\right)\right\}\frac{\partial(x)}{\partial x^n}.$$

In order that $[X_\alpha, X_\beta] \in \Omega(x)$, the coefficient of $\frac{\partial}{\partial x^n}$ in (5.4) must vanish, so that

$$(5.5)\qquad \frac{\lambda_{1\alpha}}{\lambda_{1n}}\frac{\partial}{\partial x^n}\left(\frac{\lambda_{1\beta}}{\lambda_{1n}}\right) - \frac{\partial}{\partial x^\alpha}\left(\frac{\lambda_{1\beta}}{\lambda_{1n}}\right) + \frac{\partial}{\partial x^\beta}\left(\frac{\lambda_{1\alpha}}{\lambda_{1n}}\right) - \frac{\lambda_{1\beta}}{\lambda_{1n}}\frac{\partial}{\partial x^n}\left(\frac{\lambda_{1\alpha}}{\lambda_{1n}}\right) = 0$$

[5]) See NOMIZU [4].

[6]) See BISHOP—CRITTENDEN [2].

must be fulfilled. Thus (5.5) is the necessary and sufficient condition for the integrability of $\Omega(x)$. Therefore we have

Theorem 4. a) *In an affinely connected space, whose coefficients satisfy* $H_j{}^i{}_k(x) = \lambda_{kl}(x) H_j{}^i{}_l(x)_{(*)}$ *and never vanish, there exists an* $n-1$ *dimensional differentiable distribution* $\Omega(x)$ *such that at every point* (x_0) *in the directions belonging to* $\Omega(x_0)$, *the parallel displacement is the ordinary one, being characterized by* $D\xi^i = d\xi^i = 0$.

b) *If the* $\lambda_{ij}(x)$ *satisfy the conditions* (5.5) *then through every point passes a maximal integral manifold, in this case a hypersurface* Φ, *on which the parallel displacement is the ordinary one.*

Corollary 1. In an affinely connected space, whose coefficients satisfy $H_j{}^i{}_k(x) = \lambda_{kl}(x) H_j{}^i{}_l(x)_{(*)}$ with $\lambda_{kl} = \text{const}$ and never vanish, there exists at every point a hyperplane in which the parallel displacement is the ordinary one. These hyperplanes are defined by

$$\lambda_{1i} x^i + c = 0,$$

c being a constant depending on that point x_0 through which the hyperplane passes.

Now we are able to describe a geometric feature of a tensorial connection induced by certain vector connections of type (A).

Let us consider a reducible tensorial connection induced by the linear vector connections $\overset{(a)}{H}_j{}^i{}_k(x)$ having the property

$$\overset{(a)}{H}_j{}^i{}_k(x) = \lambda_{kl}(x) \overset{(a)}{H}_j{}^i{}_l(x) \quad (*). \tag{5.6}$$

In this case we have according to (1.5) for a parallel displaced $t \in T_d$

$$Dt^{i_1 i_2 \dots i_m} = dt^{i_1 i_2 \dots i_m} + \gamma_{j_1 j_2 \dots j_m k}^{i_1 i_2 \dots i_m}(x) \xi_{(1)}^{j_1} \xi_{(2)}^{j_2} \dots \xi_{(m)}^{j_m} dx^k =$$

$$= dt^{i_1 i_2 \dots i_m} + \left\{ \sum_{a=1}^{m} \overset{(a)}{H}_{j_a}{}^{i_a}{}_k(x) \xi_{(a)}^{j_a} [\xi_{(1)}^{i_1} \dots \xi_{(a-1)}^{i_{a-1}} \xi_{(a+1)}^{i_{a+1}} \dots \xi_{(m)}^{i_m}] \right\} dx^k = 0.$$

We have seen that for $dx \in \Omega(x)$

$$\overset{(a)}{H}_{j_a}{}^{i_a}{}_k(x) \xi_{(a)}^{j_a} dx^k = 0.$$

That means that in this tensorial connection the parallel displacement of tensors $t \in T_d$ in directions belonging to $\Omega(x)$ is characterized by $Dt^\alpha = dt^\alpha = 0$. Thus we have

Theorem 5. *If a tensorial connection is induced by linear vector connections* $\overset{(a)}{H}_j{}^i{}_k(x)$ *of type* (5.6) *then in the directions belonging to the distribution* $\Omega(x)$ *the parallel displacement of the tensors* $t \in T_d$ *is characterized by* $Dt^\alpha = dt^\alpha = 0$.

Corollary 2. a) If the $\lambda_{kl}(x)$ in (5.6) satisfy (5.5) then the parallel displacements of the tensors $t \in T_d$ on the hypersurface Φ are characterized by $dt^\alpha = 0$. b) If the λ_{kl} in (5.6) are constant then (5.5) is satisfied and Φ is a hyperplane.

References

[1] A. Cossu, Nozioni generali sulle connessioni tensoriali di specie qualunque, *Rend. Mat.* e *Appl.*, **21** (1962), 167—218.
[2] R. Bishop—R. Crittenden, Geometry of manifolds *New York and London,* 1964.
[3] A. Kawaguchi, On the theory of non-linear connections I. In troduction to the theory of general non-linear connections, *Tensor, N. S.,* **2** (1952), 123—142.
[4] K. Nomizu, Lie groups and differential geometry, *Tokyo,* 1956.
[5] L. Tamássy, On non-linear tensorial connections. *Publ. Math., (Debrecen)*, **16** (1969), 193—197.

(Received 14. September, 1973.)