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Discontinuous groups in homogeneous Riemannian spaces by classification of D-symbols

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Abstract. In this paper we classify the *3-dimensional D-symbols* [4], [7], [5], [13] $(\mathcal{D},\mathcal{M})$ up to cardinality $|\mathcal{D}| = 3$ of the *vertices* of *D-diagram* $\mathcal{D}$. We describe all the possible matrix functions $\mathcal{M}$ for each D-diagram $\mathcal{D}$ such that the *combinatorial (topological) tiling* $(\mathcal{T},\Gamma)$ according to $(\mathcal{D},\mathcal{M})$, will have a *true metric realization in a simply connected homogenous Riemannian 3-space* $\mathcal{S}^3$*, one of the 8 Thurston geometries* [1], [14], [15]: $\mathbf{S}^3$, $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$, $\widetilde{\mathbf{SL_2R}}$, **Nil**, **Sol**, with a corresponding group Γ of isometries in the space $\mathcal{S}^3$. Furthermore, we describe the (generalized) orbifold [1], [2], [6], [14], $\mathcal{S}^3/\Gamma$ with the corresponding I-labelled ($I = \{0, 1, 2, 3 = d\}$) *simplicial subdivision* obtained from each D-symbol.

The phenomenon of *Thurston splitting* [2], [11], [14], [15] *along spherical (*$\mathbf{S}^2$*-) or Euclidean* (toric, $\mathbf{E}^2$-) *suborbifold* also occurs in our new classification that are summarized in Tables and Figures. As a new tool, we describe algorithms which bring our D-symbols into canonical ordered forms and list their equivalence classes by a new '$<$' relation. After having implemented our algorithms to computer we can proceed by the dimension d of space, by the increasing vertex numbers of D-diagrams as illustrated. In this paper the combinatorial and differential geometry are combined. We discuss starting results and raise open problems.

1. Introduction with examples related to Figures and Tables

We start with the familiar face-to-face cube tiling [13] $\mathcal{T} := \mathcal{T}_{cube}$ in the Euclidean space $\mathcal{S}^3 := \mathbf{E}^3$ and illustrate the procedure how to get its D-symbol $(\mathcal{D},\mathcal{M}) =: \mathcal{D}_1$ and the corresponding group $\mathbf{\Gamma_1}(x; y; z) := \mathbf{\Gamma_1}(4; 3; 4)$ in our Figures and Tables.

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Consider the *barycentric subdivision* $\mathcal{C}$ of $\mathcal{T}$. Any simplex $C := A_0A_1A_2A_3$ in $\mathcal{C}$ has vertices labelled by $I := \{0,1,2,3\}$ as follows. The vertex A_3 is the midpoint of a cube as a 3-dimensional constituent of $\mathcal{T}$, then A_2 is that of a quadrate face as an incident 2-dimensional constituent of $\mathcal{T}$, and so are A_1 and A_0 for an incident edge and vertex, respectively. In the same time we introduce the corresponding I-labels for the opposite side faces of C denoted by $s_0 : A_1A_2A_3, \dots, s_3 : A_0A_1A_2$ and the *adjacency operations*

$$\sigma_0\colon \cdots\cdots, \quad \sigma_1\colon ---, \quad \sigma_2\colon \text{———}, \quad \sigma_3\colon \sim\sim\sim\sim \tag{1.1}$$

for the simplicial subdivision $\mathcal{C}$ first. Every operation σ_i, $i \in I$, is an *involutive permutation* of $\mathcal{C}$ that orders to any C the adjacent $\sigma_i C$ along its common i-face s_i. We formally introduce the (free) *Coxeter group*

$$\Sigma_I := \big(\sigma_i,\ i \in I = \{0,1,2,3\} \ \text{—}\ \sigma_i\sigma_i =: \sigma_i^2 = 1,\ i \in I\big) \tag{1.2}$$

and its action on $\mathcal{C}$ from the left (say). The action of

$$\sigma := \sigma_{i_r}\cdots\sigma_{i_2}\sigma_{i_1} \in \Sigma_I \quad \text{on any } C \in \mathcal{C} \tag{1.3}$$

can be visualized by the *path* through the simplices

$$C,\ \sigma_{i_1}(C),\ \sigma_{i_2}\sigma_{i_1}(C){:=}\sigma_{i_2}\left(\sigma_{i_1}C\right),\dots,\sigma_{i_r}\cdots\sigma_{i_2}\sigma_{i_1}(C){=:}\sigma(C), \tag{1.4}$$

i.e. from C crossing its i_1-face, then crossing the i_2-face of $\sigma_{i_1}(C)$, $\dots$, then entering $\sigma(C)$ by formula (1.3) trough its i_r-face. The simplices of $\mathcal{C}$ can also be considered as *vertices of a dual diagram* of $\mathcal{C}$, where the σ_i-operations are indicated by connecting the vertices with i-coloured edges as (1.1) shows in the role of colours. We see on our example and require, in general, that a relation

$$\overbrace{(\sigma_j\sigma_i)\cdots(\sigma_j\sigma_i)}^{m-\text{times}}(C) =: (\sigma_j\sigma_i)^m(C) = C \tag{1.5}$$

holds for every $C \in \mathcal{C}$, $i, j \in I$ with a minimal natural number $m \in \mathbb{N}$ in (1.5), it will be denoted by

$$m =: m_{ij}(C). \tag{1.6}$$

Thus a symmetric matrix function

$$M\colon \mathcal{C} \longrightarrow \mathbb{N}_{I\times I}, \quad C \longmapsto M(C) := m_{ij}(C) \tag{1.7}$$

will be introduced which specializes the action of Σ_I defined by (1.2). For our cube tiling this matrix function is constant for every $C \in \mathcal{C}$:

$$m_{ij}(C) = \begin{pmatrix} 1 & 4 & 2 & 2 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}. \tag{1.8}$$

In general, we assume that an isometry group $\Gamma \leq \operatorname{Aut}\mathcal{T}$, like a subgroup of automorphism group of our polyhedral tiling $\mathcal{T}$, leaves invariant the combinatorial incidence structure of $\mathcal{T}$, and so Γ preserves also the barycentric subdivision $\mathcal{C}$ of $\mathcal{T}$. Our notations

$$\begin{gathered}(\sigma_i C)^\gamma = \sigma_i\left(C^\gamma\right) =: \sigma_i C^\gamma, \quad (C)^{\gamma_1\gamma_2} := \left(C^{\gamma_1}\right)^{\gamma_2} =: C^{\gamma_1\gamma_2} \\ \text{for any } C \in \mathcal{C};\ \sigma_i \in \Sigma_I \text{ and } \gamma, \gamma_1, \gamma_2 \in \Gamma\end{gathered} \tag{1.9}$$

show these facts. Γ also factorizes the matrix function M by the set $\mathcal{D}$ of Γ-orbits of $\mathcal{C}$:

$$C^\Gamma = \{C^\gamma \in \mathcal{C} : \gamma \in \Gamma\} =: D \in \mathcal{D}. \tag{1.10}$$

Thus the σ_i-operations on the set $\mathcal{D}$ of simplex orbits are induced by

$$\begin{gathered}\sigma_i : D \longmapsto \sigma_i D := \sigma_i C^\Gamma = \{\sigma_i C^\gamma \in \mathcal{C} : \gamma \in \Gamma\} \\ \text{for any } i \in I,\ C \in D \in \mathcal{D}.\end{gathered} \tag{1.11}$$

The induced matrix function

$$\begin{gathered}\mathcal{M} : \mathcal{D} \longrightarrow \mathbb{N}_{I\times I}, \quad D \longmapsto \mathcal{M}(D) := M(C) := m_{ij}(C) \\ \text{for any } C \in D \in \mathcal{D}\end{gathered} \tag{1.12}$$

and so the D-symbol $(\mathcal{D}, \mathcal{M})$ will be introduced. This pair consists of a D-diagram $\mathcal{D}$ (a set, required to be finite, together with given σ_i-operations $i \in I$) and of a matrix function $\mathcal{M}$ on $\mathcal{D}$ with some natural additional requirements as it follows later in Section 2 under formulas (2.10–13).

In the case of our cube tiling $\mathcal{T} := \mathcal{T}_{cube}$ the full isometry group $\Gamma \cong \operatorname{Aut}\mathcal{T}$ acts simply transitively on the barycentric subdivision of $\mathcal{C}$, i.e. with exactly one Γ-orbit. Thus our Figures at $\mathbf{\Gamma_1}$ show the D-diagram $\mathcal{D}_\mathbf{1}$ with 1 vertex and with four loops σ_i, $i \in I$. The matrix function under (1.8) provides the presentation of $\Gamma := \mathbf{Pm\bar{3}m}$ as a crystallographic space group no **221.** in the International Tables [8]. This presentation is given in our Tables $\mathcal{D}_\mathbf{1}$—$\mathbf{\Gamma_1}(x; y; z)$ by the generating mirror reflections m_i, $i \in I$ in the i-face of any *fixed fundamental simplex* $\mathcal{F}_\mathbf{1} = C_\mathbf{1} \in \mathcal{C}_\mathbf{1}$, and the matrix (1.8) provides the relations for $\mathbf{\Gamma_1}(x;y;z)$, now with $m_{01} := x = 4$, $m_{12} := y = 3$, $m_{23} := z = 4$ as specialized matrix entries.

Every partial D-symbol $\mathcal{D}^i$, obtained by cancelling the σ_i operation and the i-th row and i-th column of (1.8), defines the matrix function $\mathcal{M}^i$ and the partial D-symbol components $(\mathcal{D}^i, \mathcal{M}^i)$. Any of them characterizes a local 2-dimensional tiling $\mathcal{T}^i$, or more its barycentric subdivision $\mathcal{C}^i$ around a corresponding i-midpoint or around any Γ-images of this i-midpoints. Our $(\mathcal{D}^i, \mathcal{M}^i)$ describes also the stabilizer subgroup $\Gamma^i(A_i)$ of $\mathcal{T}^i$, $\mathcal{C}^i$ and of the i-midpoint A_i up to conjugacy in Γ. Now we naturally require that each Γ^i, $i \in I \setminus \{0,3\}$, is finite group for a spherical, i.e. $\mathbf{S}^2$-tiling, but any of Γ^0's or Γ^3's is permitted to be either a finite spherical group or a Euclidean crystallographic plane group for an $\mathbf{E}^2$-tiling $\mathcal{C}^0$, resp. $\mathcal{C}^3$, too. This last requirement allows ideal points (ends) for A_0's or A_3's e.g. in $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{H}^2 \times \mathbf{R}$. Since we know the plane crystallographic groups [10], [17], their fundamental domains [9], moreover their D-symbol characterization [4], [7], this will restrict our choice for the matrix function $\mathcal{M}$. That is why we have many other tilings for the same D-diagram $\mathcal{D}_\mathbf{1}$ at $\mathbf{\Gamma_1}(x;y;z)$ in the Figures and Tables. The unequalities

$$\text{(1.13.a)} \qquad \frac{\pi}{x} + \frac{\pi}{y} + \frac{\pi}{2} > \pi \quad \text{and} \quad \frac{\pi}{y} + \frac{\pi}{z} + \frac{\pi}{2} > \pi$$

for angle sum of spherical triangles, guarantee that A_3 and A_0 will be proper vertices as in the case of our Euclidean cube tiling with $\mathbf{\Gamma_1}(4;3;4)$ like a so-called Coxeter reflection group. This and any other Coxeter group [3], [16] can be described by a Coxeter diagram. Here, to the generating plane reflections m_0, m_1, m_2, m_3, as vertices, we also indicate the defining relations or mirror face angles at the edges (see in the Tables) which are well-known. For instance, $\mathbf{\Gamma_1}(x;y=2;z)$ leads to an infinite series of spherical groups and $\mathbf{S}^3$-tilings where the polyhedra of $\mathcal{T}_1$ are spherical with two x-gon faces, each of the x vertices is of valence 2, z lenses meet at any edge. $\mathbf{\Gamma_1}(3;5;3)$ leads to a regular icosahedron tiling in the hyperbolic space $\mathbf{H}^3$ where 3 icosahedra meet at each edge. $\mathbf{\Gamma_1}(3;3;6)$ serves also a $\mathbf{H}^3$-tiling by regular tetrahedra with face angles $\frac{2\pi}{6}$ and so with ideal vertices. More generally, the equalities

$$\text{(1.13.b)} \qquad \frac{\pi}{x} + \frac{\pi}{y} + \frac{\pi}{2} = \pi, \quad \text{resp.} \quad \frac{\pi}{y} + \frac{\pi}{z} + \frac{\pi}{2} = \pi$$

describe that A_3, resp. A_0 is ideal vertex. E.g. $\mathbf{\Gamma_1}(2;3;6)$ provides an $\mathbf{E}^3$-tiling with regular trigonal infinite prisms; $\mathbf{\Gamma_1}(6;3;6)$ leads to a $\mathbf{H}^3$-tiling by "regular polyhedra" ($\mathbf{H}^3$-honeycomb) whose centre and the vertices all are ideal points. This latter regular polyhedron has an inscribed horosphere. The faces are hexagons meeting 3 ones at each ideal vertex as in the Euclidean plane hexagon tiling. At any edge meet 6 such infinite polyhedra. We exclude when Γ^3 resp. Γ^0 is a $\mathbf{H}^2$-plane group, although such a polyhedron would exist in the projective embedding of $\mathbf{H}^3$ with outer centre A_3 resp. outer vertex A_0.

Since a tiling $(\mathcal{T}, \Gamma)$ and its dual $(\mathcal{T}^\star, \Gamma^\star)$ have canonically isomorphic groups $\Gamma \cong \Gamma^\star$, we enumerate only one of them. The metric existence of any $\mathbf{\Gamma_1}$-tiling rests on the eigenvalue signature of the *cosinus matrix*

$$(1.14)\qquad \begin{pmatrix} 1 & -\cos\frac{\pi}{x} & 0 & 0 \\ -\cos\frac{\pi}{x} & 1 & -\cos\frac{\pi}{y} & 0 \\ 0 & -\cos\frac{\pi}{y} & 1 & -\cos\frac{\pi}{z} \\ 0 & 0 & -\cos\frac{\pi}{z} & 1 \end{pmatrix}.$$

As it is well-known, the signature $(+\,+\,++)$ leads to $\mathbf{S}^3$-tilings, $(+\,+\,+0)$ to $\mathbf{E}^3$-tilings as in our cubic case, and $(+\,+\,+-)$ provides $\mathbf{H}^3$-tilings [1], [3], [12], [16].

Now we analogously sketch the Euclidean *rhombododecahedron tiling* $\mathcal{T}_{\mathbf{2.1}} := \mathcal{T}_{rh}$ under $\mathbf{\Gamma_{2.1}}$ in Figures and Tables. $\mathcal{T}_{rh}$ is embedded into the cube tiling in the usual way. As any rhomb lies on an edge of a cube, we illustrate only a part of this tiling with the fundamental domain $A_3A_2{}^1A_0A_1{}^2A_0$ of the full isometry group $\mathbf{\Gamma_{2.1}}(2\cdot 2;\ 3;4;3) \cong \operatorname{Aut}\mathcal{T}_{rh}$. Our procedure with the barycentric subdivision $\mathcal{C}_{rh}$ leads to 2 simplex orbits under $\mathbf{\Gamma_{2.1}}$. These are represented by ${}^1A_0A_1A_2A_3 \leftrightarrow D_1 \leftrightarrow$ 1 and ${}^2A_0A_1A_2A_3 \leftrightarrow D_2 \leftrightarrow$ 2 in the simplified D-diagram $\mathcal{D}_{\mathbf{2.1}}$ where the σ_0-operation $(\cdots\cdots)$ is indicated but the loops for σ_1-, σ_2- and σ_3-operations are not (as also later on).

Our general procedure leads to the presentation of $\mathbf{\Gamma_{2.1}}$ again as a Coxeter group with 4 generating reflections $m_1, m_{1'}, m_2, m_3$ indicated only by their subindices in their simplified Coxeter diagram. Here the order $x = y = 3$ of product m_2m_3 expresses that 3 rhombododecahedra meet at any edges ${}^1A_0A_1{}^2A_0$; $v = 3$ and $w = 4$ indicate the edge-valences of the non-equivalent vertices ${}^1A_0$ resp. ${}^2A_0$ on the surfaces of rhombododecahedron. The most "interesting" information is expressed by $2\cdot u$ in the denotation $\mathbf{\Gamma_{2.1}}(2u;\ v,w;\ x=y)$, $u=2$ in our case. That means the rhomb is a quadrangle: $2\cdot u$-gon (of course), $u=2$ refers to the order of reflection-product $m_1m_{1'}$ (now $s_1 \perp s_{1'}$). The coefficient 2 refers to the number of vertices in the partial diagram only with σ_0- and σ_1-operations

$$(1.15)\qquad \mathcal{D}_{01} : \ 1 \cdots\cdots 2 \quad \text{i.e.}\quad (\sigma_1\sigma_0)^2 D_z = D_z, \quad z = 1,2.$$

So, this refers to the exponent $r_{01} = 2$ in the $\langle \sigma_0, \sigma_1 \rangle$-orbit of $\mathcal{D}_{\mathbf{2.1}}$ under the action of Σ_I. Furthermore, all these also refer to the entry $m_{01} = r_{01} \cdot u = 2 \cdot u = 4$ in the matrix function

$$\mathcal{M}\colon \mathcal{D}_{\mathbf{2.1}} \longrightarrow \mathbb{N}_{I \times I},$$

$$(1.16) \qquad D_1 \longmapsto \begin{pmatrix} 1 & 2u & 2 & 2 \\ 2u & 1 & v & 2 \\ 2 & v & 1 & x \\ 2 & 2 & x & 1 \end{pmatrix}, \qquad D_2 \longmapsto \begin{pmatrix} 1 & 2u & 2 & 2 \\ 2u & 1 & w & 2 \\ 2 & w & 1 & y \\ 2 & 2 & y & 1 \end{pmatrix}$$

$$\text{with } (u; v; w;\ x = y) = (2; 3; 4; 3)$$

as our Tables say in a shorter form.

The same D-diagram $\mathcal{D}_{\mathbf{2.1}}$ with $(u; v; w;\ x = y) = (2; 3; 5; 3)$ describes a $\mathbf{H}^3$-tiling with proper vertices, but e.g. $(2; 3; 6; 3)$ gives us a $\mathbf{H}^3$-tiling with A_3 and ${}^2A_0$ as ideal vertices at the absolute, while ${}^1A_0$ is proper vertex. Again, the *cosinus matrix*

$$(1.17) \qquad \begin{array}{c} ① \\ ①' \\ ② \\ ③ \end{array} \begin{pmatrix} 1 & -\cos\frac{\pi}{u} & -\cos\frac{\pi}{v} & 0 \\ -\cos\frac{\pi}{u} & 1 & -\cos\frac{\pi}{w} & 0 \\ -\cos\frac{\pi}{v} & -\cos\frac{\pi}{w} & 1 & -\cos\frac{\pi}{x} \\ 0 & 0 & -\cos\frac{\pi}{x} & 1 \end{pmatrix},$$

according to the simplified Coxeter diagram (Fig. $\mathcal{D}_{\mathbf{2.1}}$–$\Gamma_{\mathbf{2.1}}$), by its signature, decides the existence of metric realization in $\mathbf{S}^3$ $(+\,+\,++)$, $\mathbf{E}^3$ $(+\,+\,+0)$, $\mathbf{H}^3$ $(+\,+\,+-)$. The principal $3{\times}3$ minors determine the qualities of vertices. For instance, the signature of $(1, 1', 2)$-minor tells us the quality of vertex A_3. For $(u, v, w) = (2, 3, 4)$ or $(2, 3, 5)$ the signature is $(+\,+\,+)$, that means A_3 is proper. But for $(u, v, w) = (2, 3, 6)$ that is $(+\,+\,0)$ indicating an $\mathbf{E}^2$-stabilizer for A_3, i.e. A_3 is ideal vertex. The signatures of $(1, 2, 3)$- and $(1', 2, 3)$-minors characterize ${}^2A_0$ and ${}^1A_0$, respectively.

Our Tables $\mathcal{D}_{\mathbf{2.1}}$–$\Gamma_{\mathbf{2.1}}$ tell that $\mathbf{\Gamma}_{\mathbf{2.1}} \cong \operatorname{Aut}\mathcal{T}_{\mathbf{2.1}}$ maximal group iff $v < w$, else $\mathbf{\Gamma}_{\mathbf{1}}(\bar{x} = 2u;\ \bar{y} = v = w;\ \bar{z} = x = y)$ is a supergroup, preserving the combinatorial structure of the corresponding tiling $\mathcal{T}_{\mathbf{2.1}}$. Indeed, in case $v = w$ $\mathcal{T}_{\mathbf{2.1}}$ has a richer automorphism group. Namely, the combinatorial reflection (as the operation $\sigma_0 : D_1 \mapsto D_2 = \sigma_0 D_1$ dictates)

$$(1.18) \qquad m_0 : {}^1A_0A_1A_2A_3 \longmapsto {}^2A_0A_1A_2A_3 \quad \text{of} \quad \mathcal{F}_{\mathbf{2.1}} = {}^1A_0{}^2A_0A_2A_3$$

(the fundamental domain of $\mathbf{\Gamma_{2.1}}(2u;\ v,w;\ x=y)$ in case $v=w$) can be extended to the entire tiling $\mathcal{T}_{\mathbf{2.1}}$. In the language of D-symbols we say: there is a *D-morphism,* i.e. a surjection

$$\Psi : \mathcal{D}_{\mathbf{2.1}} \longrightarrow \mathcal{D}_{\mathbf{1}} \quad \text{with} \quad (\sigma_i D)^{\Psi} = \sigma_i(D^{\Psi}), \quad \mathcal{M}(D^{\Psi}) = \mathcal{M}(D) \quad \text{for any } D \in \mathcal{D}_{\mathbf{2.1}},\ i \in I, \tag{1.19}$$

preserving the σ_i-operations and the matrix function with the corresponding parameters (the bars, if occur, only distinguish the letters in the two symbols). Then $\mathcal{T}_{\mathbf{2.1}}$ can be derived from $\mathcal{T}_{\mathbf{1}}$ by *symmetry breaking* [4], [5], [7], [13].

Our Tab. $\mathcal{D}_{\mathbf{2.1}}$–$\Gamma_{\mathbf{2.1}}$ tells that $\mathbf{\Gamma_{2.1}}$ is optimally presented by the tiling $\mathcal{T}_{2.1}$, iff — in addition to the other conditions — $3 \le v, x$; else $\mathbf{\Gamma_1}$ and $\mathcal{T}_{\mathbf{1}}$ — with appropriate parameters (not detailed) — provide a simpler presentation for the group $\mathbf{\Gamma_{2.1}}$. Indeed, if $u=1$, then $v=w$ would lead to non-maximal groups; similarly $3=u=v=w$, when A_3 is ideal vertex with $\mathbf{E}^2$-stabilizer. If $v=2$ or $x=2$ then the Coxeter diagram reduces to that of $\mathbf{\Gamma_1}$.

Our Tab. $\mathcal{D}_{\mathbf{2.1}}$–$\Gamma_{\mathbf{2.1}}$ allows the condition $v \le w$ by logical symmetry. Indeed, if $v > w$ holds, we change $1 \leftrightarrow 2$ in the notations of vertices of $\mathcal{D}_{\mathbf{2.1}}$, to obtain our case.

After this detailed introduction we discuss for $d=3$ more concisely the inverse problem: For each D-diagram $\mathcal{D}$ of a classification list up to cardinality $|\mathcal{D}|=3$ we give the possible matrix functions $\mathcal{M}$ so that each D-symbol $(\mathcal{D},\mathcal{M})$ shall be realizable, first by a combinatorial tiling $(\mathcal{T},\Gamma)$ in a simply connected topological 3-space $\mathcal{S}^3$. In this way $(\mathcal{D},\mathcal{M})$ will represent a (generalized) good orbifold [1], [2], [6], [14] $\mathcal{O}^3(\mathcal{X}^3,\mathcal{A})$, i.e. a topological 3-space $\mathcal{X}^3$ with a compatibile atlas $\mathcal{A}$, where each point P has a neighbourhood $\mathcal{U}_P$ in $\mathcal{A}$ that is homeomorphic to $\mathbf{R}^3$ factorized by a finite group G_O fixing the origin $O \in \mathbf{R}^3$, corresponding to $P \in \mathcal{X}^3$. Our natural generalization allows finitely many "ideal" points in $\mathcal{X}^3$, any of them has neighbourhood homeomorphic to $(\mathbf{R}^2/G)\times\mathbf{R}$ where G is a Euclidean plane crystallographic group that acts on $\mathbf{R}^3$, extended along a fixed plane $\mathbf{R}^2$, preserving its halfspaces. The "ideal" point, considered in $\mathcal{X}^3$, corresponds to the common ideal point $+\infty$ of $\mathbf{R}$-fibers of $(\mathbf{R}^2/G)\times\mathbf{R}$ *(embedded in the projective sphere* $\mathbf{PS}^3$ *of* $\mathbf{R}^3$*, but it is not important now [12], [16]).* Each orbifold and tiling $(\mathcal{T},\Gamma)$ will be given by a canonical fundamental domain $\mathcal{F}_\Gamma$ as the D-symbol dictates by our later Algorithm 2.3. $\mathcal{F}_\Gamma$ is endowed by an involutive face pairing $\mathcal{I}$ (identifications) which generates the group Γ. The symbol $(\mathcal{D},\mathcal{M})$ provides also the defining relations for Γ.

Altough the general criteria for isometric realizations have not been completely determined yet *(these are related with the Thurston conjecture [1], [2], [14], [15], see our necessary assumption and conjecture at Alg. 2.3.e and papers [12], [13])*, second we give also the metric realization for each D-symbol $(\mathcal{D}, \mathcal{M})$ and tiling $(\mathcal{T}, \Gamma)$, if exists, in a Thurston geometry from the list in the Abstract. If such a metric realization does not exist, i.e. our good orbifold is non-geometric [6], [11] then we give the corresponding $\mathbf{S}^2$- and $\mathbf{E}^2$-suborbifolds, respectively, and splittings along them, according to the Thurston's orbifold conjecture. All the results are summarized in the Figures and Tables as indicated above by the starting examples.

2. On classification of D-diagrams and D-symbols, in general

Definition 2.1 of a D-diagram (to honour of B. N. Delone (Delaunay), M. S. Delaney and A. W. M. Dress [4], [5], [7], [13]): Let $\mathcal{D} := (\Sigma_I, \mathcal{D})$ a finite set endowed by $d+1$ involutive permutations as *σ_i-operations $i \in I = \{0, 1, \dots, d\}$ generating the left hand side action of a free Coxeter group Σ_I by (1.2) which is transitive on $\mathcal{D}$.* Particularly, think of dimension $d = 3$, and use the conventions (1.1) in Sect. 1. Any element $D_z \leftrightarrow z$ of $\mathcal{D}$ can be considered as a vertex of an I-coloured graph, called *D-diagram $\mathcal{D}$,* or more visually, a vertex corresponds to an $\mathbf{R}$-coordinatized I-labelled simplex $(D_z; \Delta_I)$ as a Cartesian product with the standard simplex

$$\Delta_I = \left\{ \mathbf{x} := (x^0, x^1, \dots, x^d) \in \mathbf{R}^I : 0 \le x^k \text{ for any } k \in I \text{ and } \sum_{k=0}^{d} x^k = 1 \right\}$$

$$\text{(2.1)} \qquad \text{its } i\text{-facets } \Delta_I^i := \big\{ \mathbf{x} \in \Delta_I : x^i = 0 \big\} \text{ and } i\text{-vertices}$$
$$A_i\big(a_i^j\big) \in \Delta_I \text{ by } a_i^j = \delta_i^j \text{ the Kronecker symbol } i, j \in I.$$

Here the simplex Δ_I *wears the usual affine topology.* $\mathcal{D}$ is assumed to have the *discrete topology.* We imagine as many simplices $(D_1; \Delta_I)$, ... , $(D_n; \Delta_I)$ as many elements $\mathcal{D}$ has, $n := |\mathcal{D}|$ denotes the cardinality of $\mathcal{D}$. That means, we can introduce the *standard topological realization,* denoted by $\mathrm{Top}(\Sigma_I, \mathcal{D}) := (\mathcal{D}; \Delta_I)/\sim$, as the Cartesian product with pointwise identifications $\sim$: $(D; \mathbf{y}) \sim (\sigma_i D; \mathbf{y})$ for every $D \in \mathcal{D}$ and $\mathbf{y} \in \Delta_I^i$, $i \in I$. $\square$

As a tool for visualization, we may introduce the *“local reflection”*

$$\sigma_i : (D;\mathbf{x}) \longmapsto \sigma_i(D;\mathbf{x}) := (\sigma_i D;\, \mathbf{x}) \text{ for every } \mathbf{x} \in \Delta_I \tag{2.2}$$

in the facet $(D;\Delta_I^i) \sim (\sigma_i D;\, \Delta_I^i)$, fixed pointwise under σ_i for a “local” $D \in \mathcal{D}$, $i \in I$. Then we can define and visualize the action of Σ_I on $\mathcal{D}$ analogously as under formulas (1.3–4). Furthermore we can introduce

Definition 2.2, the action of the (extended) fundamental group of $\mathrm{Top}(\Sigma_I,\, \mathcal{D}(D_1))$

$$\pi_1(\Sigma_I,\, \mathcal{D}(D_1)) := \Sigma_I(D_1) := \{\sigma \in \Sigma_I : \sigma(D_1) = D_1\} \tag{2.3}$$

is well-defined *up to a conjugacy* in Σ_I related to a starting element $D_1 \in \mathcal{D}$. □

Indeed, if $D = \varrho D_1$ is an arbitrary vertex of the Σ_I-connected $\mathcal{D}$ with $\varrho \in \Sigma_I$, then $\Sigma_I(D) = \varrho \Sigma_I(D_1)\varrho^{-1}$.

Definition 2.3. The universal covering space $\widetilde{\mathrm{Top}}(\Sigma_I,\mathcal{D}) := (\Sigma_I;\mathcal{D};\Delta_I)/\!\sim$ *of* $\mathrm{Top}(\Sigma_I,\mathcal{D})$ can be defined, first again, as a Cartesian product $(\Sigma_I;\mathcal{D};\Delta_I)$ with the discrete topology of $(\Sigma_I;\mathcal{D})$ and the usual affine topology on Δ_I. Second again, we introduce identifications on $(\Sigma_I;\mathcal{D};\Delta_I)$

$$\begin{gathered} \sim\colon (\varrho\sigma;\, D_1;\mathbf{x}) \sim (\varrho\sigma\varrho^{-1};\, D;\mathbf{x}) \\ \text{and} \quad (\varrho\sigma\varrho^{-1};\, D;\mathbf{y}) \sim (\sigma_i\varrho\sigma\varrho^{-1};\, D;\mathbf{y}) \\ \text{for any } D_1 = \sigma D_1,\ D = \varrho D_1 \text{ in } (\Sigma_I;\mathcal{D}), \\ \mathbf{x} \in \Delta_I,\ \mathbf{y} \in \Delta_I^i,\ i \in I. \end{gathered} \tag{2.4}$$
□

If $\sigma D_1 = D_1 = \tau D_1$, then $\sigma\tau D_1 = D_1$ also holds, i.e. $\sigma,\tau,\sigma\tau \in \Sigma_I(D_1)$. To $\Sigma_I(D_1)$ we correspond *the group* $\tilde{\Sigma}_I(\mathcal{D})$ *of covering transformations of* $\widetilde{\mathrm{Top}}(\Sigma_I,\mathcal{D})$ which acts from the right, written exponentially, and preserves the Σ_I-action on simplices of $\widetilde{\mathrm{Top}}(\Sigma_I,\mathcal{D})$ as

$$(\sigma_i D)^{\tilde\sigma} = \sigma_i(D^{\tilde\sigma}) =: \sigma_i D^{\tilde\sigma}, \quad (D)^{\tilde\sigma\tilde\tau} := \big(D^{\tilde\sigma}\big)^{\tilde\tau} = D^{\widetilde{\sigma\tau}} \tag{2.5}$$

will denote. Indeed, we define for $D := (1;D;\Delta_I) \sim (\varrho;D_1;\Delta_I) =: \varrho D_1$ in $\widetilde{\mathrm{Top}}(\Sigma_I,\mathcal{D})$

$$D^{\tilde\sigma} := (\varrho D_1)^{\tilde\sigma} := \varrho\sigma D_1 = \varrho\sigma\varrho^{-1}D \quad \text{iff } \sigma D_1 = D_1 \text{ in } (\Sigma_I,\mathcal{D}). \tag{2.6}$$

Then

$$(\sigma_i D)^{\tilde\sigma} := (\sigma_i\varrho D_1)^{\tilde\sigma} := (\sigma_i\varrho)\sigma D_1 = \sigma_i\varrho\sigma\varrho^{-1}D = \sigma_i\big(D^{\tilde\sigma}\big)$$

hold, indeed. Moreover,

$$\left(D^{\tilde{\sigma}}\right)^{\tilde{\tau}} := (\varrho\sigma D_1)^{\tilde{\tau}} := (\varrho\sigma)\tau D_1 = \varrho(\sigma\tau)\varrho^{-1}D =: D^{\widetilde{\sigma\tau}}$$

show also the *homomorphism* Φ *of the fundamental group* $\Sigma_I(D_1)$ *of* $\mathrm{Top}(\Sigma_I, \mathcal{D}(D_1))$ *onto the covering transformation group* $\tilde{\Sigma}_I(\mathcal{D})$ *of* $\widetilde{\mathrm{Top}}(\Sigma_I, \mathcal{D})$ by

$$\Phi : \Sigma_I(D_1) \longrightarrow \tilde{\Sigma}_I(\mathcal{D}), \quad \sigma \longmapsto \left\{\varrho\sigma\varrho^{-1} : \varrho \in \Sigma_i\right\} =: \tilde{\sigma}. \tag{2.7}$$

A standard argumentation shows that our Φ *is even an isomorphism.* Here we do not prove this fact, only mention that it is a byproduct from a more general construction for universal covering of a good orbifold [1], [11], [14].

Definition 2.4. The D-diagram $(\Sigma_I, \mathcal{D})$ automaticly induces its *subdiagram components* ${}^c(\Sigma_J, \mathcal{D})$, $J \subset I$, which characterize lower-dimensional parts of $\mathrm{Top}(\Sigma_I, \mathcal{D})$. Thus, the components ${}^c\mathcal{D}_{ij}$, $i < j \in I$ with their σ_i- and σ_j-operations define a symmetric matrix function with entries from the natural numbers $\mathbb{N}$:

$$\begin{aligned} &\mathcal{R}\colon \mathcal{D} \longrightarrow \mathbb{N}_{I\times I}; \quad D \longmapsto \mathrm{r}_{ij}(D), \quad i, j \in I \\ &\text{by} \quad \mathrm{r}_{ij}(D) := \min\left\{r \in \mathbb{N} : (\sigma_j\sigma_i)^r(D) = D\right\} \end{aligned} \tag{2.8}$$

with the following requirements:

$$\begin{aligned} &\mathrm{r}_{ii}(D) = 1; \quad \mathrm{r}_{ij}(D) = \mathrm{r}_{ji}(D) = \mathrm{r}_{ji}(\sigma_i D); \\ &\mathrm{r}_{ij}(D) \in \{1, 2\} \quad \text{if} \quad 1 < j - i \\ &\text{for any } D \in \mathcal{D},\ i < j \in I. \qquad \square \end{aligned} \tag{2.9}$$

Definition 2.5 of a D-symbol $(\mathcal{D}, \mathcal{M})$. A D-diagram $(\Sigma_I, \mathcal{D})$ together with a matrix function, with entries from $\mathbb{N}^\infty := \mathbb{N} \cup \infty$:

$$\mathcal{M} : \mathcal{D} \longrightarrow \mathbb{N}^\infty_{I\times I}; \quad D \longmapsto \mathcal{M}(D) := \mathrm{m}_{ij}(D) := \mathrm{r}_{ij}(D) \cdot \mathrm{v}_{ij}(D) \tag{2.10}$$

with $v_{ij}(D) \in \mathbb{N}^\infty$ as *rotational orders* or *branching numbers,* is called a D-symbol (Delone–Delaney–Dress-symbol), if the following requirements are fulfilled:

$$\mathrm{m}_{ii}(D) = 1; \quad \mathrm{m}_{ij}(D) = \mathrm{m}_{ji}(D) = \mathrm{m}_{ji}(\sigma_i D); \tag{2.11}$$

$$\mathrm{m}_{ij}(D) = 2 \quad \text{if} \quad 1 < j - i; \quad \mathrm{m}_{ij}(D) \geq 2 \quad \text{if} \quad 1 = j - i; \tag{2.12}$$

$$(\sigma_j\sigma_i)^{\mathrm{m}_{ij}(D)}(D) = D \quad \text{if} \quad \mathrm{m}_{ij} \neq \infty \tag{2.13}$$

for any $D \in \mathcal{D}$ and $i < j \in I$. $\square$

From our present investigations $m_{ij}(D) = \infty = v_{ij}(D)$ will be excluded. In case $d = 3$ we have required proper $(d-2)$-faces, i.e. edges for the simplices in the barycentric subdivision $\mathcal{C}$ of a tiling $\mathcal{T}$ being described later. Of course (2.10) provides dependences among the requirements for the functions r_{ij}, v_{ij}, m_{ij}. These are constant on any component ${}^c\mathcal{D}_{ij}$ of $(\mathcal{D}, \mathcal{M})$, $i < j \in I$.

Definition 2.6 of realization. To a given D-symbol $(\mathcal{D}, \mathcal{M})$ we want to construct a simply connected d-space $\mathcal{S}^d$, moreover a tiling $\mathcal{T}$ in $\mathcal{S}^d$ with a barycentric subdivision $\mathcal{C}$ of I-labelled simplices. The left action of Σ_I should be transitive on $\mathcal{C}$ and compatible with the right action of a group $\Gamma \leq \operatorname{Aut} \mathcal{T}$. We require that the orbit space $\mathcal{C}/\Gamma$ and the Σ_I action, induced on it, is just isomorphic to $\mathcal{D} := (\Sigma_I, \mathcal{D})$. Furthermore, we also require that for any $C \in D \in \mathcal{C}/\Gamma = \mathcal{D}$ the matrix function $\mathcal{M}$ provides the minimal exponent $m_{ij}(D) =: m$ so that $(\sigma_j\sigma_i)^m(C) = C$ holds for any $i, j \in I$. *If such a construction exists, then we call it a topological (orbifold) realization of D-symbol $(\mathcal{D}, \mathcal{M})$.* Analogously we can define the other realizations, e.g. like metric realization in a space of constant curvature with group Γ of isometries, or in other spaces. *The existence of such a realization is questionable, in general.* $\square$

In our papers [12], [13] *we proposed a procedure for such a construction, in general.* The basic idea was that the D-symbol $(\mathcal{D}, \mathcal{M})$ itself dictates how to glue a fundamental domain $\mathcal{F}$ from $|\mathcal{D}| =: n$ simplices, and how to pair the free facets of $\mathcal{F}$ to generate a group Γ and a fundamental tiling $\langle \mathcal{F}, \Gamma \rangle$ with Γ-images of $\mathcal{F}$. The free $(d-2)$-face classes of $\mathcal{F}$, by means of the matrix function $\mathcal{M}$, tell us how to glue the Γ-images of $\mathcal{F}$ at the $(d-2)$-faces. Moreover, they tell us the defining relations for the face pairing generators of Γ by so-called *Poincaré–Aleksandrov algorithm* [1], [12], [16]. Moreover, the tiling $\langle \mathcal{F}, \Gamma \rangle$ just defines a simply connected space $\mathcal{S}^d$, if the fundamental domain $\mathcal{F}$ is nice enough, e.g. the interior of its each k-face is homeomorphic to an open k-simplex $(k \in I)$. The group Γ, however, may collapse by the consequences of the defining relations above. This is the case, e.g. if one from the four types of 2-dimensional bad orbifolds [11], [12], [14] does occur among the partial D-symbol components of $(\mathcal{D}, \mathcal{M})$ with any 3 colours from I (see Alg. 2.3.e, (2.16)). Then the rotational orders v_{ij} and so m_{ij} should be reduced according to (2.10).

Our classification will show these phenomena on concrete examples.

Now we give a general scheme for classification of D-diagrams and D-symbols. This is our new initiative in this paper, although this type of method is well-known in combinatorics.

1. We define an *ordered form for each D-symbol* $(\Sigma_I,\, \mathcal{D}(D_1),\, \mathcal{M})$ *with starting element* $D_1 \leftrightarrow$ 1 by numbering the other elements. So we can define a *distance between* D_1 *and* D_k as $D_1D_k = k-1$. Choosing another starting element, a (non-symmetric) *distance function* can be defined in the whole D-diagram, independent of $\mathcal{M}$.
2. We define a '$<$' *relation between any two D-symbols* each with distinguished starting element.
3. On the base of **1**–**2** we define a *smallest numbering* for a fixed D-symbol to choose a *representative* from isomorphic variants.
4. Comparing the representatives, we *list the D-symbols increasingly.*
5. To each D-symbol $(\mathcal{D}, \mathcal{M})$ in the list we determine its *automorphism group* $\mathrm{Aut}(\mathcal{D}, \mathcal{M})$ by the step **3**. That is not trivial iff we have more smallest numberings. We take the orbits of $\mathrm{Aut}(\mathcal{D}, \mathcal{M})$ as elements from a smaller D-symbol $(\mathcal{D}^n, \mathcal{M})$ with the induced Σ_I-action and the same matrix function. This is called *the normalizer D-symbol of* $(\mathcal{D}, \mathcal{M})$.
6. As usual [4], [5], [7], [13] we introduce to each D-symbol $(\mathcal{D}, \mathcal{M})$ *its smallest D-morphic image* $(\mathcal{D}^\star, \mathcal{M}^\star)$ if there is a surjective mapping

$$\begin{gathered}\Psi\colon \mathcal{D} \longrightarrow \mathcal{D}^\star, \quad D \longmapsto D^\Psi \\ \text{with} \quad (\sigma_i D)^\Psi = \sigma_i(D^\Psi) \quad \text{and} \quad \mathcal{M}(D) = \mathcal{M}^\star(D^\Psi) \\ \text{for each } D \in \mathcal{D},\ i \in I.\end{gathered} \tag{2.14}$$

   This characterizes isomorphic tilings $\mathcal{T} \cong \mathcal{T}^\star$ with groups $\Gamma \le \Gamma^\star \cong \mathrm{Aut}\,\mathcal{T}$, the maximal group for $\mathcal{T} \cong \mathcal{T}^\star$. If Ψ above is bijective, then it is a *D-isomorphism.* If $|\mathcal{D}^\star| < |\mathcal{D}|$ stands for the cardinalities, then $(\mathcal{T}, \Gamma)$ *is called a symmetry breaking of* $(\mathcal{T}^\star, \Gamma^\star)$ according to the D-symbols, respectively.
7. We arrange the D-symbols into topological families. Each family is represented by the common smallest D-morphic image, i.e. by the maximal group $\Gamma = \mathrm{Aut}\,\mathcal{T}$.

Algorithm 2.1. Let $(\Sigma_I,\, \mathcal{D}(D_1))$ be the D-diagram of a D-symbol with a starting element $D_1 \in \mathcal{D}$. We number the other elements (vertices) of $\mathcal{D}$ by the Σ_I-operations according to the natural increasing ordering of $I = \{0, 1, 2, \ldots, d\}$ as follows:

a) Assume, we have already numbered the elements $D_1, \ldots, D_r$, $r < |\mathcal{D}| =: n$. Consider $\sigma_0(D_r)$, $\sigma_1(D_r)$. The first of them, not listed yet, will be D_{r+1} if exists.

b) Else we take $\sigma_2(D_r), \dots, \sigma_2(D_1)$; $\dots$; $\sigma_d(D_r), \dots, \sigma_d(D_1)$. The first new one will be D_{r+1}.

c) Then we proceed with $r \to r+1$ as above. Since Σ_I acts transitively on the finite $\mathcal{D}$, we end at D_n, $n = |\mathcal{D}|$.

d) The distance of any two elements D_x, D_y can be obtained: We choose $D_x = D_{1'}$ for starting element and proceed as above. If we get $D_y = D_{k'}$ then the distance is $D_x D_y = k-1$. □

Algorithm 2.2. Let $(\Sigma_I, \mathcal{D}(D_1), \mathcal{M}) = \mathcal{D}$ and $(\Sigma'_{I'}, \mathcal{D}'(D_{1'}), \mathcal{M}') =: \mathcal{D}'$ be two D-symbols each with distinguished starting element. We define $\mathcal{D} < \mathcal{D}'$ by the following preferences **a**–**d**:

a) $|I| < |I'|$ ($|\cdot|$ denotes cardinality); preference of dimension.

b) If both dimensions $=: d$ then $|\mathcal{D}| < |\mathcal{D}'|$; preference of cardinality.

c) If both cardinaties $=: n$ then we compare distances in $\mathcal{D}$ and $\mathcal{D}'$, respectively. Consider equally numbered elements and their σ_i-images in reverse preference in $I = I'$:
- $D_1\sigma_d D_1 < D_{1'}\sigma_d D_{1'}$; if '=' holds then $D_2\sigma_d D_2 < D_{2'}\sigma_d D_{2'}$; $\dots$; if '=' then $D_n\sigma_d D_n < D_{n'}\sigma_d D_{n'}$; of course, it is enough to go till $n-1$;
- if '=' then $D_1\sigma_{d-1} D_1 < D_{1'}\sigma_{d-1} D_{1'}$; $\dots$; if '=' then $D_n\sigma_{d-1} D_n < D_{n'}\sigma_{d-1} D_{n'}$;

$\vdots$
- if '=' then $D_1\sigma_0 D_1 < D_{1'}\sigma_0 D_{1'}$; $\dots$; if '=' then $D_n\sigma_0 D_n < D_{n'}\sigma_0 D_{n'}$.

d) If '=' stands in each place before, then the D-diagrams are isomorphic. Then come the matrix functions by increasing preferences in their $01, 12, \dots, (d-1)d$ entries for the equal components of $\mathcal{D}_{01}$, $\mathcal{D}_{12}, \dots, \mathcal{D}_{(d-1)d}$:
- $m_{01}(D_1) < m'_{01}(D_{1'})$; if '=' then $m_{01}(D_2) < m'_{01}(D_{2'})$; $\dots$; if '=' then $m_{01}(D_n) < m'_{01}(D_{n'})$; of course, we compare whole 01 orbits, as later on, too;
- if '=' then $m_{12}(D_1) < m'_{12}(D_{1'})$; $\dots$; if '=' then $m_{12}(D_n) < m'_{12}(D_{n'})$;

$\vdots$
- if '=' then $m_{(d-1)d}(D_1) < m'_{(d-1)d}(D_{1'})$; $\dots$; if '=' then $m_{(d-1)d}(D_n) < m'_{(d-1)d}(D_{n'})$.

e) If '=' stands in each place before, then the two D-symbols clearly are D-isomorphic and lie in the same equivalence class. □

Proposition 2.1. *Our '$<$' relation is trichotom and transitive on the equivalence classes of D-symbols with distinguished starting elements.*

The proof *is obvious.* In each place we compare natural numbers, within zero, whose ordering satisfies these properties. □

We remark that other preferences are also possible in Alg. 2.1.a or in Alg. 2.2.c: These would lead to other orderings, considered less natural for our reason. Our preferences can be applied for generating our list of D-symbols systematically. We do not give further details since we plan another publication about this topic.

Algorithm 2.3 of constructing canonical fundamental domain $\mathcal{F}$ and the tiling $(\mathcal{T}, \Gamma)$ for a D-symbol. Consider $\mathcal{D} := (\Sigma_I, \mathcal{D}(D_1), \mathcal{M})$ by its smallest numbering. We proceed by Alg. 2.1: Glue $D_2 = \sigma_{i_1}(D_1)$ to D_1 by the first non-trivial σ_{i_1}-operation in Alg. 2.1.a, i.e. form $(D_1; \Delta_I) \cup (D_2; \Delta_I)$ identified along $(D_1; \Delta_I^{i_1}) \sim (D_2; \Delta_I^{i_1})$ and form a convex affine chart to copy them in $\mathbf{R}^d$.

- **a**) Assume, we have already glued $(D_1; \Delta_I) \cup \cdots \cup (D_r; \Delta_I)$ by the corresponding $\sigma_{i_1}-, \ldots, \sigma_{i_{r-1}}$-operations and imagine a convex "affine chart" in $\mathbf{R}^d$ as follows. D_{r+1} join D_r along their free i_r-facet, i.e. D_r is not σ_{i_r}-related to the former simplices, and we keep convexity. (The convexity of a metric realization of $\mathcal{F}$ is not guaranteed yet, in general.)
- **b**) Else all facets of D_r are not free, either glued (covered) or paired by former facets. Any pair provides either a generating transformation or the identity for the group Γ. This also depends on the matrix function $\mathcal{M}$, namely on the rotational orders v_{ij} in (2.10) (cf. the Alg. 6.1 in our paper [12]).
- **c**) At the end we have $\mathcal{F}$ glued of $n := |\mathcal{D}|$ simplices, its interior is homeomorphic to an open d-simplex. The paired facets of $\mathcal{F}$ provide a complete system $\mathcal{I}$ of generators for Γ. The $(d-2)$-faces of simplices in $\mathcal{F}$ are either covered, i.e. they are surrounded by facets as a consequence of $\mathcal{M}$; or they form Γ-equivalence classes, and $\mathcal{M}$ implies for each class a defining relation for Γ by the Poincaré–Aleksandrov algorithm [12], as we indicated after Def. 2.6.
- **d**) Consider the partial D-symbol $(\Sigma_{I\setminus\{k\}}, \mathcal{D}^k, \mathcal{M}^k)$ obtained from $\mathcal{D}$ by deleting the σ_k-operation and by restricting $\mathcal{M}$, $k \in I$. Then $\mathcal{D}^k$ falls into connected components. Any component ${}^c\mathcal{D}^k$ describes a $(d-1)$-dimensional tiling $(\mathcal{T}^k, \Gamma^k)$ and its simplicial subdivision around the

k-labelled midpoint of the tiling $\mathcal{T}$ and around the Γ-images of this k-midpoint. Thus the components of $\mathcal{D}^k$, $k \in I$, give a complete combinatorial description of our tiling $(\mathcal{T}, \Gamma)$ for $\mathcal{D}$, if the existence is guaranteed (cf. Def. 2.6).

e) Consider any partial D-symbol $(\Sigma_{ijk}, \mathcal{D}_{ijk}, \mathcal{M}_{ijk}) =: \mathcal{D}_{ijk}$ obtained from $\mathcal{D}$ by keeping the σ_i-, σ_j-, σ_k-operations and deleting the others. Moreover, we restrict the matrix function $\mathcal{M}$ on entries m_{ij}, m_{ik}, m_{jk} for any $i < j < k \in I$. Think of $d = 3$ where steps **d** and **e** coincide! Any connected component ${}^c\mathcal{D}_{ijk}$ is a 2-dimensional D-symbol, and it may determine a corresponding tiling $({}^c\mathcal{T}_{ijk}, {}^c\Gamma_{ijk})$ and a simply connected covering 2-surface $\mathcal{S}^2$ dually round the $(d-3)$-simplex at the intersection of corresponding i-facets, j-facets and k-facets according to our topological realization $\mathrm{Top}(\Sigma_I, \mathcal{D})$ in Def. 2.1.
We know that our tiling $({}^c\mathcal{T}_{ijk}, {}^c\Gamma_{ijk})$ may be equivariant to a spherical $(\mathbf{S}^2)$, Euclidean $(\mathbf{E}^2)$, hyperbolic $(\mathbf{H}^2)$ tiling if the curvature [4]

$$K({}^c\mathcal{D}_{ijk}) = \sum_{D \in {}^c\mathcal{D}_{ijk}} \left(\frac{1}{m_{ij}(D)} + \frac{1}{m_{ik}(D)} + \frac{1}{m_{jk}(D)} - 1 \right) \begin{matrix} > \\ = \\ < \end{matrix} 0, \tag{2.15}$$

respectively. Moreover, this condition is also satisfactory if in the first ('$>$') case the four types of bad orbifolds are excluded. We give them by the signature [9], [10], [17] (see also [4], [7]):

$$\begin{aligned} &(+, 0;\ [u];\ \{\ \}), && 1 < u; \\ &(+, 0;\ [u, v];\ \{\ \}), && 1 < u < v; \\ &(+, 0;\ [\ ];\ \{(u)\}), && 1 < u; \\ &(+, 0;\ [\ ];\ \{(u, v)\}), && 1 < u < v. \qquad \square \end{aligned} \tag{2.16}$$

Our general conjecture is that any D-symbol $(\mathcal{D}, \mathcal{M})$ determines its topological realization $(\mathcal{T}, \Gamma)$ in a simply connected space $\mathcal{S}^d$ if the fundamental domain $\mathcal{F}$ of $(\mathcal{D}, \mathcal{M})$, with its facet pairings and presentation by Alg. 2.3, represents a good orbifold, i.e., if any 2-dimensional "suborbifold induced by $(\mathcal{D}, \mathcal{M})$" is a good 2-orbifold.

Although the last part of this conjecture is "folklore" by an analogous Thurston's conjecture for good orbifolds, the author intends to give a complete formulation and proof, since it seems to be not published yet, in general. In our case $d = 3$ the proper l-vertices will be characterized by (2.15) if $\{i, j, k, l\} = \{0, 1, 2, 3\} = I$ and $K({}^c\mathcal{D}^l) > 0$ if (2.16) is excluded. If $l = 0$ or 3 then $K({}^c\mathcal{D}^l) = 0$ is also allowed for ideal 0-vertex and

3-centre as indicated in Sect. 1. The curvature $K({}^c\mathcal{D}^l) < 0$ is excluded from our investigation now. But the above subsymbols do not describe the 2-suborbifolds of $(\mathcal{D}, \mathcal{M})$ yet. The examination of the fundamental domain by $(\mathcal{D}, \mathcal{M})$ provides still a local method for investigating these 2-suborbifolds. Section 3 will show the difficulties of such investigations which are not algorithmized yet (see also [12]).

We do not describe the 8 Thurston geometries, listed in the Abstract, since in our classification $d = 3$, till $|\mathcal{D}| =: n = 3$ only few of them occur [1], [14], [15]. We suggest to the reader for studying again our introductory examples in Sect. 1. Then turn to Sect. 3 and to Figures and Tables. We shall elaborate the case at $\mathcal{D}_{\mathbf{3.2}}$–$\Gamma_{\mathbf{3.2}}$ in details. see also the corresponding Fig. 3.2. The other cases will be described more sketchily.

3. Classification of D-symbols and their optimal realizations, $d = 3$, $1 \leq |\mathcal{D}| \leq 3$

In the introduction we have already discussed the series of matrix functions belonging to the D-diagram of 1 vertex with 4 loops Fig. $\mathbf{\Gamma_1}$ and Tab. $\mathbf{\Gamma_1}$ show the phenomena. The other cases will be enumerated in similar manner, so as $\mathbf{\Gamma_{2.1}}$ where $2 \leq x = y$ just refer to the situation mentioned in Alg. 2.3.e. Namely, our Tab. $\mathcal{D}_{\mathbf{2.1}}$—$\mathbf{\Gamma_{2.1}}$ contains a date

$$\Gamma^1(A_1) = (+, 0; [\ ];\ \{(x, y)\}) \Longrightarrow x = y \tag{3.1}$$

else the partial diagram $\mathcal{D}_{023} = \mathcal{D}^1$ with $\mathcal{M}^1 : 2 \leq x \neq y$ would lead to the partial tiling $(\mathcal{T}^1, \Gamma^1)$ around the 1-midpoint A_1, where reflections m_2 and m_3 act on a sphere $\mathbf{S}^2$ with different dihedral orders $x \neq y$ at the opposite poles which is impossible. This would be the fourth type of bad orbifolds in (2.16).

Now we run through the 15 classes of D-diagrams with 2 elements (vertices).

$\mathcal{D}_{\mathbf{2.2}}$—$\mathbf{\Gamma_{2.2}}$ leads to 5 reflections, indicated at the simplified Coxeter diagram in Fig. $\mathbf{\Gamma_{2.2}}$. We briefly describe every matrix function $\mathcal{M}$, analogous to (1.16). For instance $m_{01}(D_1) = m_{01}(D_2) = 2u$ in $\mathcal{D}_{\mathbf{2.2}}$—$\mathbf{\Gamma_{2.2}}$ describes the reflections m_0 and m_0' at A_2A_3 by the relation $(m_0m_0')^u$. The signature of $\Gamma^3(A_3)$, or the curvature formula by (2.15) with $(2 \leq u, v)$ for optimal cases yield

$$K(\mathcal{D}_{012}) = \left(\frac{1}{2u} + \frac{1}{2v} - \frac{1}{2}\right) + \left(\frac{1}{2u} + \frac{1}{2v} - \frac{1}{2}\right) = \frac{1}{u} + \frac{1}{v} - 1 \geq 0 \tag{3.2}$$

$$\text{if } u = v = 2, \text{ then } K(\mathcal{D}_{012}) = 0,$$

i.e. $\Gamma^3(A_3)$ is an $\mathbf{E}^2$-group and A_3 is an ideal point. In Tab. $\mathcal{D}_{\mathbf{2.2}}$—$\mathbf{\Gamma_{2.2}}$ we find the series of $\mathbf{H}^2\times\mathbf{R}$ realizations $(u; v; w; x) = (2; 2; 2; x)$ depending on $x \geq 3$. Indeed, Fig. $\mathbf{\Gamma_{2.2}}$ show the triangle $A_3A_2{}^2A_1$ with angles 0, $\frac{\pi}{2}$, $\frac{\pi}{x}$ realizable in $\mathbf{H}^2$ and the "prism" over it in direction $\mathbf{R}$ with $A_3A_2{}^2A_1$-congruent "parallel sections", e.g. $A_3{}^1A_1A_0$. To the $\mathbf{H}^3$-realizations we start with the proper or ideal vertex-domain A_0 with face angles $\frac{\pi}{v} = \frac{\pi}{2}$, $\frac{\pi}{w}$, $\frac{\pi}{x}$ at the rays A_0A_3, $A_0{}^1A_1$, $A_0{}^2A_1$, respectively. The plane ${}^1A_1A_2A_3$ is orthogonal to the ray $A_0{}^1A_1$ and hyperbolic parallel with the ray A_0A_3. This determines the place of ${}^1A_1$ uniquely. Similarly, we get ${}^2A_1$, then A_2. In the ideal point A_3 there meet 4 planes with 3 rectangles at A_0A_3, ${}^1A_1A_3$, ${}^2A_1A_3$, then we have perpendicular faces also at A_2A_3, and we are done with the construction in $\mathbf{H}^3$.

Fig. and Tab. $\mathcal{D}_{\mathbf{2.3}}$—$\mathbf{\Gamma_{2.3}}$ show that $m_{23}(D_1) = w = x = m_{23}(D_2)$ is necessary to a topological realization, again by $\Gamma^1(A_1)$ and (2.16.). But then $\mathbf{\Gamma_1}(\bar{x} = u;\ \bar{y} = 2v;\ \bar{z} = w)$ is a normalizer supergroup of $\mathbf{\Gamma_{2.3}}$ (bars only distinguish the parameters of the two group series). We have as many $\mathbf{S}^3$-, $\mathbf{H}^3$-realization as the supergroup series tell us, we have the only degenerate $\mathbf{E}^3$-realizations with A_0 and A_0' as Γ-equivalent ideal vertices.

"$\mathcal{D}_{\mathbf{2.4}}$—$\mathbf{\Gamma_{2.4}}$ is dual to $\mathcal{D}_{\mathbf{2.2}}$—$\mathbf{\Gamma_{2.2}}$" means that if we change $\sigma_0 \to \bar{\sigma}_3$, $\sigma_1 \to \bar{\sigma}_2$, $\sigma_2 \to \bar{\sigma}_1$, $\sigma_3 \to \bar{\sigma}_0$ in D-diagram $\mathcal{D}_{\mathbf{2.2}}$ and the corresponding matrix entries $m_{01} \leftrightarrow m_{23}$, $m_{12} \leftrightarrow m_{12}$ then we get D-symbol $\mathcal{D}_{\mathbf{2.4}}$. The group $\mathbf{\Gamma_{2.4}}$ is equivariantly isomorphic to $\mathbf{\Gamma_{2.2}}$, the equivariance is defined by the duality above.

$\mathcal{D}_{\mathbf{2.5}}$—$\mathbf{\Gamma_{2.5}}$ leads to non-optimal groups and tilings with D-normalizer supergroups $\mathbf{\Gamma_1}$ by parameters $\bar{x} = 2u$, $\bar{y} = 2v$, $\bar{z} = 2w$. Thus only $u = 1$, $v = 2$, $w = 2$ leads to a degenerate $\mathbf{E}^3$-realization according to our assumptions.

$\mathcal{D}_{\mathbf{2.6}}$—$\mathbf{\Gamma_{2.6}}$ leads to non-optimal groups and tilings again. $\mathcal{D}_{\mathbf{2.6}}$ is a self-dual diagram, so $m_{01} := 2u \leq 2w =: m_{23}$ can be assumed to obtain non-equivariant groups. From $\mathbf{\Gamma_1}(\bar{x} = 2u;\ \bar{y} = v;\ \bar{z} = 2w)$ we derive the corresponding metric tilings, e.g. $u = 2$, $v = 3$, $w = 3$ yield the marked cube tiling in $\mathbf{E}^3$ with the crystallographic group $\mathbf{Pm\bar{3}}$.

$\mathcal{D}_{\mathbf{2.7}}$—$\mathbf{\Gamma_{2.7}}$ yields again the D-normalizer supergroups $\mathbf{\Gamma_1}(\bar{x} = u;\ \bar{y} = v;\ \bar{z} = 2w)$, and e.g. the $\mathbf{E}^3$-tiling with marked cubes under the group **226.Fm$\mathbf{\bar{3}}$c**.

$\mathcal{D}_{\mathbf{2.8}}$—$\mathbf{\Gamma_{2.8}}$ leads to dually equivariant tilings to $\mathcal{D}_{\mathbf{2.1}}$—$\mathbf{\Gamma_{2.1}}$.

$\mathcal{D}_{\mathbf{2.9}}$—$\mathbf{\Gamma_{2.9}}$ leads to self-dual optimal tilings if $v \leq w$. Now $u \leq x$ is assumed. The reflection subgroups, with Coxeter diagram pictured there, are well-known [16]. These imply our cases, among them 2 Euclidean ones.

The cases $\mathcal{D}_{\mathbf{2}.10}$—$\mathbf{\Gamma}_{\mathbf{2}.10}$, … , $\mathcal{D}_{\mathbf{2}.14}$—$\mathbf{\Gamma}_{\mathbf{2}.14}$ lead to analogous tilings.

$\mathcal{D}_{\mathbf{2}.15}$—$\mathbf{\Gamma}_{\mathbf{2}.15}$ leads to self-dual non-optimal tilings with D-normalizer supergroups $\mathbf{\Gamma_1}(u; v; w)$ just with the same parameters. An $\mathbf{E}^3$-tiling is the cube tiling under **207.P432**.

Next we consider D-diagrams with 3 vertices. The first, by Alg. 2.2,

$\mathcal{D}_{\mathbf{3.1}}$—$\mathbf{\Gamma}_{\mathbf{3.1}}$ yields Coxeter groups with diagrams in our picture. So again, $s = t$ is a necessary condition for topological realization in Tab. $\mathcal{D}_{\mathbf{3.1}}$—$\mathbf{\Gamma}_{\mathbf{3.1}}$. We find that our assumptions exclude the optimal realizations.

$\mathcal{D}_{\mathbf{3.2}}$—$\mathbf{\Gamma}_{\mathbf{3.2}}$ *is our most interesting self-dual case*. The fundamental domain $\mathcal{F}$ consist of 3 glued simplices with 6 free reflection facets, at most. Our assumptions allow this maximal number of reflections for parameters $(p; q; r; s; t) = (2; 2; 2; 2; 2)$ where

$$\Gamma^3(A_3) = (+, 0; [\ ];\ \{(r, 2, p, q)\}) \tag{3.3}$$

is the stabilizer of A_3, and

$$\Gamma^0(A_0) = (+, 0; [\ ];\ \{(r, 2, t, s)\}) \tag{3.4}$$

the stabilizer of A_0 are just $\mathbf{E}^2$-groups. This Coxeter diagram and the corresponding last picture show that non-isometric realizations occur in the 8 Thurston geometries. The reason is the suborbifold $\mathbf{E}^2/\mathbf{pmm}$ whose groups are generated by 4 reflections

$$\begin{gathered} m_0' : {}^2A_1{}^2A_2A_3{}^1A_2, \quad m_1 : {}^2A_2A_3A_0, \\ m_2 : A_3A_0{}^1A_1, \quad m_3 : A_0{}^1A_1{}^1A_2{}^2A_1 \\ \text{with} \quad (m_0'm_1)^2 = (m_1m_2)^2 = (m_2m_3)^2 = (m_3m_0')^2 = 1. \end{gathered} \tag{3.5}$$

The corresponding surface is transversally placed in the middle of $\mathcal{F}$ up to equivariant isotopy. Splitting our $\mathcal{F}$ along this $\mathbf{E}^2$-suborbifold [2], we get two pieces, both can wear the metric of a $\mathbf{H}^2\times\mathbf{R}$-orbifold as the last picture shows. Each $\mathbf{H}^2$-component is a Lambert quadrangle with ideal vertex (at A_3 and A_0, respectively) and 3 rectangles.

We have 5 generating reflections if $p = 1$, i.e. $m_0 = m_0'$, or $r = 1$, i.e. $m_1 = m_2$ as our specialized Coxeter diagrams indicate in the second row of our picture page $\mathbf{\Gamma}_{\mathbf{3.2}}$.

Our case $p = 1;\ 1 < r, t, s$ with non-$\mathbf{H}^2$-stabilizer for A_0, implies $s = 2 = r = t$, and the earlier $\mathcal{D}_{\mathbf{2.2}}$—$\mathbf{\Gamma}_{\mathbf{2.2}}$ ($\bar{u} = t = 2;\ \bar{v} = r = 2;\ \bar{w} = 2;\ \bar{x} = q$) provides simpler presentation. The choice $p = 1,\ \ t = 1$ implies $\mathbf{\Gamma_1}$ again.

The case $r = 1$ (see the corresponding Coxeter diagram and Fig. $\mathbf{\Gamma_{3.2}}$ in the 3rd row) *leads to the most interesting optimal cases.* An exceptional non-maximal group $\mathbf{\Gamma_{3.2}}(2p = 4 = q;\ 3r = 3;\ s = 4 = 2t)$ is just the $\mathbf{E}^3$ space group **123.P4/mmm** to the cubic tiling with maximal supergroup $\mathbf{\Gamma_1}(4;3;4)$. The $\mathbf{E}^3$-optimal realization with $\mathbf{\Gamma_{3.2}}(2\cdot 2,\ 3;\ 3\cdot 1;\ 6,\ 2\cdot 2) \cong$ **P6/mmm** leads to a Euclidean tiling with regular trigonal prisms.

In general, $r = 1,\ \ p = 2 = t$ leads to prismatic tilings

$$\text{(3.6)} \qquad \text{in } \mathbf{S}^2{\times}\mathbf{R}, \text{ if } \ \frac{1}{q} + \frac{1}{s} > \frac{1}{2}; \qquad \text{in } \mathbf{H}^2{\times}\mathbf{R}, \text{ if } \ \frac{1}{q} + \frac{1}{s} < \frac{1}{2}.$$

Non-geometric good orbifolds occur in the following cases

$$\text{(3.7)} \qquad \begin{array}{c} r = 1 < p,\ 3 \le t \text{ and } \ \dfrac{1}{q} + \dfrac{1}{s} > \dfrac{1}{2}, \quad \text{i.e. } \mathbf{S}^2 - \text{splittings} \\ \text{or} \quad \dfrac{1}{q} + \dfrac{1}{s} = \dfrac{1}{2} \quad \text{i.e. } \mathbf{E}^2 - \text{splittings} \end{array}$$

are possible in the "middle" of our fundamental domain $\mathcal{F}$. The two splitted parts of $\mathcal{F}$ have the Coxeter diagrams as our Fig. $\mathbf{\Gamma_{3.2}}$ in the 3rd row shows. Each $\mathbf{S}^2$-splitting becomes a proper point of the corresponding part. Each $\mathbf{E}^2$-splitting becomes an ideal point. The Coxeter diagrams serve us the isometric realizations, may be different for the two parts.

The cases

$$\text{(3.8)} \qquad r = 1 < p,\ 3 \le t \text{ and } \ \frac{1}{q} + \frac{1}{s} < \frac{1}{2}, \quad \text{i.e. } \mathbf{H}^2 - \text{suborbifolds}$$

characterize $\mathbf{H}^3$-realizations, may be not optimal. For instance $\mathbf{\Gamma_{3.2}}(2p = 4 = q;\ 3r = 3;\ s = 6 = 2t)$ leads to the maximal $\mathbf{H}^3$-group $\mathbf{\Gamma_1}(x = 4;\ y = 3;\ z = 6)$ and tiling $\mathcal{T}_1$ of cubes with proper A_3 and ideal A_0. $\mathbf{\Gamma_{3.2}}(2p = 4 = q;\ 3r = 3;\ s = 5,\ 2t = 6)$ leads to a nice maximal $\mathbf{H}^3$-tiling $\mathcal{T}$ with $\mathbf{\Gamma_{3.2}} = \operatorname{Aut}\mathcal{T}$. The tiles are cubes with 1 Γ-class of proper vertices but 2 Γ-classes of faces and edges, the latter ones are surrounded by $m_{12}(D_1) = s = 5$ and $m_{12}(D_2) = m_{12}(D_3) = 2t = 6$ neighbours, respectively.

$\mathcal{D}_{\mathbf{3.3}}$—$\mathbf{\Gamma_{3.3}}$ leads to maximal tilings and groups if $r \neq 2s$. Our Figures and Tables show the cases in details. From $\Gamma^3(A_3)$ we see that only $(p;q) = (2;1)$ and $(1;2)$ provide optimal cases with ideal centre A_3. $\Gamma^0(A_0, A_0')$ shows many values of parameters when they are proper or ideal vertices. Our pictures indicate the fundamental domain $\mathcal{F}$ and its r-image about ${}^2A_1A_3$ together, so that we see a domain whose reflection images fil the space. Again, we have obtained 2 infinite series of optimal tilings in $\mathbf{H}^2{\times}\mathbf{R}$

according to running s (left hand side) resp. r (right hand side picture), the $\mathbf{R}$-direction is A_0A_0' resp. $A_0{}^1A_1$. We have interesting $\mathbf{H}^3$-tilings, optimal and non-optimal, too. An optimal $\mathbf{H}^3$-series is $(p;q;r;s) = (2;1;r;2)$ $3 \leq r$. But the case $r = 4$ is not maximal because $r = 2s$. Then the maximal $\mathbf{\Gamma_1}(\bar{x} = 3p = 6;\ \bar{y} = 3q = 3;\ \bar{z} = r = 2s = 4)$ provides a regular hyperbolic tiling with "horospherical" solid: A_3 is ideal, 3 hexagons meet in each proper vertex, any edge is surrounded 4 solids (the edges fall into 2 classes under the original $\mathbf{\Gamma_{3.3}}$).

Remark 1. For $d = 3$, $|\mathcal{D}| =: n = 4$ we have 82 non-isomorphic D-diagrams listed by Alg. 2.2 for another publication. An important open problem is, how many non-isomorphic D-diagrams exist for a fixed dimension d, and fixed $n := |\mathcal{D}|$. Give an estimate, e.g. a good upper bound, at least.

Remark 2. For any space group in $\mathbf{E}^3$ the minimal D-symbol seems to be very important. For $\mathbf{E}^2$ and $\mathbf{S}^2$ this is simple and solved. For $\mathbf{E}^3$ we have determined the minimal D-symbol in cases of many space groups, we are working on this problem.

Remark 3. While having prepared this paper, I have learned that O. Delgado Friedrichs: Euclidicity criteria for three-dimensional branched triangulations, Ph. D. Dissertation, Bielefeld, 1994 – among other results – enumerated all D-symbols up to cardinality $|\mathcal{D}| = 10$ which have $\mathbf{E}^3$-realizations.

Remark 4. The recent paper "Higher toroidal regular polytopes" by P. McMullen and E. Schulte, *Advances in Math.* **117** (1996), 17–51 determines nearly all regular toroids for each dimension. This corresponds to D-symbols on 1 node where $(\mathcal{T}, \Gamma)$ is realized on the Euclidean n-torus. For this information and other useful advices I would like to thank the Referees.

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Tables

$\mathcal{D}_{\mathbf{1}} - \mathbf{\Gamma_1}(x;y;z)$ $2 \le x,y,z$ – *Sd:* $x \le z$ – $\mathbf{\Gamma_1}$ is maximal • $m_0\colon A_1A_2A_3$, $m_1\colon A_2A_3A_0$, $m_2\colon A_3A_0A_1$, $m_3\colon A_0A_1A_2$ – m_0^2, m_1^2, m_2^2, m_3^2; $(m_0m_1)^x$, $(m_0m_2)^2$, $(m_0m_3)^2$; $(m_1m_2)^y$, $(m_1m_3)^2$; $(m_2m_3)^z$ •
$\Gamma^3(A_3) = (+,0;[\,];\{(x,2,y)\})$ – (x,y)– $\mathbf{S}^2$: $(2,y)$, $(3,3)$, $(3,4)$, $(3,5)$ – $\mathbf{E}^2$: $(3,6)$, $(4,4)$ – $\mathbf{H}^2$: else ∘ $\Gamma^0(A_0) = (+,0;[\,];\{(y,2,z)\})$ – (y,z)– $\mathbf{S}^2$: $(2,z)$, $(3,3)$, $(3,4)$, $(3,5)$ – $\mathbf{E}^2$: $(3,6)$, $(4,4)$ – $\mathbf{H}^2$: else • $(x;y;z)$– $\mathbf{S}^3$: $(x;2;z)$, $(2;3;3)$, $(2;3;4)$, $(2;3;5)$; $(2;4;3)$, $(2;5;3)$; $(3;3;3)$, $(3;3;4)$, $(3;3;5)$, $(3;4;3)$ – $\mathbf{E}^3$: $(4;3;4)$ **221.Pm$\bar{\mathbf{3}}$m** / $(2;3;6)$, $(2;4;4)$, $(2;6;3)$ A_0 id. v. – $\mathbf{H}^3$: $(3;5;3)$, $(4;3;5)$, $(5;3;5)$ proper vertices / $(3;3;6)$, $(3;4;4)$, $(4;3;6)$, $(5;3;6)$, A_0 id. v. / $(3;6;3)$, $(4;4;4)$, $(6;3;6)$ A_3 and A_0 ideal vertices / else outer vertex occurs □

$\mathcal{D}_{\mathbf{2.1}}$—$\mathbf{\Gamma_{2.1}}(2u;\ v,w;\ x=y)$ $1 \le u$; $2 \le v \le w$; $2 \le x$ – $\mathbf{\Gamma_{2.1}}$ is max. iff $v < w$; else $\mathbf{\Gamma_1}(2u;\ v=w;\ x=y)$ is supergroup – $\mathbf{\Gamma_{2.1}}$ is optimal iff, in addition, $3 \le v,x$; else $\mathbf{\Gamma_1}$ provides simpler presentation. •
$m_1\colon A_2A_3{}^1A_0$, $m_1'\colon A_2A_3{}^2A_0$, $m_2\colon A_3{}^1A_0{}^2A_0$, $m_3\colon {}^1A_0{}^2A_0A_2$ – m_1^2, ${m_1'}^2$, m_2^2, m_3^2; $(m_1m_1')^u$, $(m_1m_2)^v$, $(m_1'm_2)^w$, $(m_1m_3)^2$, $(m_1'm_3)^2$; $(m_2m_3)^x$ •
$\Gamma^3(A_3) = (+,0;[\,];\{(u,v,w)\}) \Rightarrow 2 \le u$ – (u,v,w)– $\mathbf{S}^2$: $(2,3,4)$, $(2,3,5)$ – $\mathbf{E}^2$: $(2,3,6)$ – $\mathbf{H}^2$: else ∘ $\Gamma^1(A_1) = (+,0;[\,];\{(x,y)\}) \Rightarrow x = y$ ∘ $\Gamma^0({}^1A_0) = (+,0;[\,];\{(v,2,x)\})$ – (v,x)– $\mathbf{S}^2$: $(3,3)$, $(3,4)$, $(3,5)$ – $\mathbf{E}^2$: $(3,6)$, $(4,4)$ – $\mathbf{H}^2$: else ∘ $\Gamma^0({}^2A_0) = (+,0;[\,];\{(w,2,y)\})$ – (w,y)– $\mathbf{S}^2$: $(3,3)$; $(3,4)$, $(3,5)$ – $\mathbf{E}^2$: $(3,6)$, $(4,4)$ – $\mathbf{H}^2$: else • $(u;v;w;x)$– $\mathbf{E}^3$: $(2;3;4;3)$ **225.Fm$\bar{\mathbf{3}}$m** – $\mathbf{H}^3$: $(2;3;5;3)$ pr. v. / $(2;3;4;4)$ ${}^2A_0$ id. v. / $(2;3;6;3)$ A_3 and ${}^2A_0$ id. v. □

$\mathcal{D}_{\mathbf{2.2}}$—$\mathbf{\Gamma_{2.2}}(2u;\ 2v;\ w,x)$ $1 \le u,v$; $2 \le w \le x$ – $\mathbf{\Gamma_{2.2}}$ is max. iff $w < x$; else $\mathbf{\Gamma_1}$ provides simpler presentation • $m_0\colon {}^1A_1A_2A_3$, $m_0'\colon {}^2A_1A_2A_3$, $m_2\colon A_3A_0{}^1A_1$, $m_2'\colon A_3A_0{}^2A_1$, $m_3\colon A_0{}^1A_1A_2{}^2A_1$ – m_0^2, ${m_0'}^2$, m_2^2, ${m_2'}^2$, m_3^2; $(m_0m_0')^u$, $(m_0m_2)^2$, $(m_0'm_2')^2$ $(m_0m_3)^2$, $(m_0'm_3)^2$, $(m_2m_2')^v$, $(m_2m_3)^w$, $(m_2'm_3)^x$ • $\Gamma^3(A_3) = (+,0;[\,];\{(u,2,v,2)\}) \Rightarrow u = v = 2$ – $(u;v)$– $\mathbf{E}^2$: $(2;2)$ – $\mathbf{H}^2$: else ∘ $\Gamma^0(A_0) = (+,0;[\,];\{(v,w,x)\})$ – $(v;w;x)$– $\mathbf{S}^2$: $(2;2;x)$; $(2;3;4)$, $(2;3;5)$ – $\mathbf{E}^2$: $(2;3;6)$ – $\mathbf{H}^2$: else • $(u;v;w;x)$– $\mathbf{H}^2{\times}\mathbf{R}$: $(2;2;2;x)$ $3 \le x$, A_3 id. v. – $\mathbf{H}^3$: $(2;2;3;4)$, $(2;2;3;5)$ A_3 id. v. / $(2;2;3;6)$ A_3 and A_0 id. v. □

$\mathcal{D}_{\mathbf{2.3}}$—$\mathbf{\Gamma_{2.3}}(u^+;\ 2v;\ w=x)$ $1 \le v$; $2 \le u,w$ – $\mathbf{\Gamma_{2.3}}$ is not maximal, $\mathbf{\Gamma_1}(u;\ 2v;\ w)$ is supergroup • $r\colon A_2A_3A_0 \to A_2A_3A_0'$, $m_2\colon A_3A_0A_0'$,

$m_3\colon A_0A_0'A_2$ – r^u, m_2^2, m_3^2; $(m_2rm_2r^{-1})^v$, $m_3rm_3r^{-1}$, $(m_2m_3)^w$ • $(u;v;w)$– $\mathbf{S}^3$ – $\mathbf{E}^3$: $(1;2;4)$, $(1;3;3)$ (A_0,A_0') id. v. – $\mathbf{H}^3$ □

$\mathcal{D}_{\mathbf{2.4}}$—$\mathbf{\Gamma_{2.4}}$ is dual to $\mathcal{D}_{\mathbf{2.2}}$—$\mathbf{\Gamma_{2.2}}$ □

$\mathcal{D}_{\mathbf{2.5}}$—$\mathbf{\Gamma_{2.5}}(2u;\ 2v;\ 2w)$ $1 \le u,v,w$ – $\mathbf{\Gamma_{2.5}}$ is not max., $\mathbf{\Gamma_1}(2u;\ 2v;\ 2w)$ is supergroup • $m_1\colon A_2A_3A_0$, $m_1'\colon A_2A_3A_0'$, $r\colon A_3A_0A_1 \to A_3A_0'A_1$, $m_3\colon A_0A_0'A_2$ – m_1^2, ${m_1'}^2$, r^2, m_3^2; $(m_1m_1')^u$, $(m_1rm_1'r)^v$, $(m_1m_3)^2$, $(m_1'm_3)^2$ $(m_3rm_3r)^w$ • $(u;v;w)$– $\mathbf{S}^3$ – $\mathbf{E}^3$: $(1;2;2)$ (A_0A_0') id. v. – $\mathbf{H}^3$ □

$\mathcal{D}_{\mathbf{2.6}}$—$\mathbf{\Gamma_{2.6}}(2u;\ v^+;\ 2w)$ $1 \le u,w$; $2 \le v$ – *Sd:* $u \le w$ – $\mathbf{\Gamma_{2.6}}$ is not max., $\mathbf{\Gamma_1}(2u;\ v;\ 2w)$ is supergroup • $m_0\colon A_1A_2A_3$, $m_0'\colon A_1'A_2A_3$, $r\colon A_3A_0A_1 \to A_3A_0A_1'$, $m_3\colon A_0A_1A_2A_1'$ – m_0^2, ${m_0'}^2$, r^v, m_3^2; $(m_0m_0')^u$, $m_0rm_0'r^{-1}$, $(m_0m_3)^2$, $(m_0'm_3)^2$, $(m_3rm_3r^{-1})^w$ • $(u;v;w)$– $\mathbf{S}^3$ – $\mathbf{E}^3$: $(2;3;2)$ **200.Pm$\bar{\mathbf{3}}$** / $(1;3;3)$, $(1;4;2)$ (A_0,A_0') id. v. – $\mathbf{H}^3$ □

$\mathcal{D}_{\mathbf{2.7}}$—$\mathbf{\Gamma_{2.7}}(u^+;\ v^+;\ 2w)$ $1 \le w$; $2 \le u,v$ – $\mathbf{\Gamma_{2.7}}$ is not max., $\mathbf{\Gamma_1}(u;\ v;\ 2w)$ is supergroup • $r_1\colon A_2A_3A_0 \to A_2A_3A_0'$, $r_2\colon A_3A_0A_1 \to A_3A_0'A_1$, $m_3\colon A_0A_0'A_2$ – r_1^u, r_2^2, m_3^2; $(r_1r_2)^v$, $m_3r_1m_3r_1^{-1}$, $(m_3r_2m_3r_2)^w$ • $(u;v;w)$– $\mathbf{S}^3$ – $\mathbf{E}^3$: $(4;3;2)$ **226.Fm$\bar{\mathbf{3}}$c** / $(2;3;3)$, $(2;4;2)$ (A_0,A_0') id. v. – $\mathbf{H}^3$ □

$\mathcal{D}_{\mathbf{2.8}}$—$\mathbf{\Gamma_{2.8}}$ is dual to $\mathcal{D}_{\mathbf{2.1}}$—$\mathbf{\Gamma_{2.1}}$ □

$\mathcal{D}_{\mathbf{2.9}}$—$\mathbf{\Gamma_{2.9}}(2u;\ v,w;\ 2x)$ $1 \le u,x$; $2 \le v \le w$ – *Sd:* $u \le x$ – $\mathbf{\Gamma_{2.9}}$ is max. iff $v < w$, else $\mathbf{\Gamma_1}(2u;\ v = w;\ 2x)$ is supergroup – $\mathbf{\Gamma_{2.9}}$ is not opt. if $u = 1$, then $v = w$ and $\mathbf{\Gamma_1}(2;v;\ 2x)$ is supergroup • $m_1\colon A_2A_3A_0$, $m_1'\colon A_2A_3A_0'$, $m_2\colon A_3A_0A_0'$, $r\colon A_0A_1A_2 \to A_0'A_1A_2$ – m_1^2, ${m_1'}^2$, m_2^2, r^2; $(m_1m_1')^u$, $(m_1m_2)^v$, $(m_1'm_2)^w$ $m_1rm_1'r$, $(m_2rm_2r)^x$ • $\Gamma^3(A_3) = (+,0;[\,];\{(u,v,w)\})$ $\Rightarrow 2 \le u$ – (u,v,w)– $\mathbf{S}^2$: $(2,2,w)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ – $\mathbf{E}^2$: $(2,3,6)$, $(2,4,4)$ – $\mathbf{H}^2$: else ◦
$\Gamma^0(A_0,A_0') = (+,0;[\,];\{(v,w,x)\})$ $\Rightarrow 2 \le x$ – (v,w,x)– $\mathbf{S}^2$: $(2,2,x)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ – $\mathbf{E}^2$: $(2,3,6)$, $(2,4,4)$ – $\mathbf{H}^2$: else •
$(u;v;w;x)$– $\mathbf{S}^3$: $(2;2;w;2)$ $3 \le w$, $(2;2;3;3)$, $(2;2;3;4)$ – $\mathbf{E}^3$: $(2;2;4;3)$ **229.Im$\bar{\mathbf{3}}$m**, $(3;2;3;3)$ **227.Fd$\bar{\mathbf{3}}$m** – $\mathbf{H}^3$: $(2;2;3;5)$, $(2;2;5;3)$, $(2;3;4;2)$, $(2;3;5;2)$, $(3;2;3;4)$, $(3;2;3;5)$, $(3;2;4;3)$, $(3;2;5;3)$, $(4;2;3;4)$, $(4;2;3;5)$, $(5;2;3;5)$ pr. v. / $(2;2;3;6)$, $(2;2;4;4)$, $(2;2;6;3)$, $(3;2;3;6)$, $(3;2;4;4)$, $(4;2;3;6)$, $(5;2;3;6)$ (A_0,A_0') id. v. / $(2;3;6;2)$, $(3;2;6;3)$, $(4;2;4;4)$, $(6;2;3;6)$ A_3, (A_0,A_0') id. v. / else out. v. □

$\mathcal{D}_{\mathbf{2}.10}$—$\mathbf{\Gamma_{2.10}}$ is dual to $\mathcal{D}_{\mathbf{2.5}}$—$\mathbf{\Gamma_{2.5}}$ □

$\mathcal{D}_{\mathbf{2}.11}$—$\mathbf{\Gamma_{2.11}}$ is dual to $\mathcal{D}_{\mathbf{2.3}}$—$\mathbf{\Gamma_{2.3}}$ □

$\mathcal{D}_{\mathbf{2}.12}$—$\mathbf{\Gamma_{2.12}}(u^+;\ 2v;\ 2w)$ $1 \leq v, w$; $2 \leq u$ – $\mathbf{\Gamma_{2.12}}$ is not max., $\mathbf{\Gamma_1}(u,\ 2v,\ 2w)$ is supergroup • $r_1\colon A_2A_3A_0 \to A_2A_3A_0'$, $m_2\colon A_3A_0A_0'$, $r_3\colon A_0A_1A_2 \to A_0'A_1A_2$ – r_1^u, m_2^2, r_3^2; $(m_2r_1m_2r_1^{-1})^v$, $(r_1r_3)^2$, $(m_2r_3m_2r_3)^w$ • $(u;v;w)$– $\mathbf{S}^3$ – $\mathbf{E}^3$: $(2;2;2)$ (A_0, A_0') id. v. – $\mathbf{H}^3$ □

$\mathcal{D}_{\mathbf{2}.13}$—$\mathbf{\Gamma_{2.13}}$ is dual to $\mathcal{D}_{\mathbf{2}.12}$—$\mathbf{\Gamma_{2.12}}$ □

$\mathcal{D}_{\mathbf{2}.14}$—$\mathbf{\Gamma_{2.14}}$ is dual to $\mathcal{D}_{\mathbf{2.7}}$—$\mathbf{\Gamma_{2.7}}$ □

$\mathcal{D}_{\mathbf{2}.15}$—$\mathbf{\Gamma_{2.15}}(u^+;\ v^+;\ w^+)$ $2 \leq u, v, w$ – *Sd:* $u \leq w$ – $\mathbf{\Gamma_{2.15}}$ is not max., $\mathbf{\Gamma_1}(u;v;w)$ is supergroup • $r_1\colon A_2A_3A_0 \to A_2A_3A_0'$, $r_2\colon A_3A_0A_1 \to A_3A_0'A_1$, $r_3\colon A_0A_1A_2 \to A_0'A_1A_2$ – r_1^u, r_2^2, r_3^2; $(r_1r_2)^v$, $(r_1r_3)^2$, $(r_2r_3)^w$ • $(u;v;w)$– $\mathbf{S}^3$ – $\mathbf{E}^3$: $(4;3;4)$ **207.P432** / $(2;3;6)$, $(2;4;4)$ $(2;6;3)$ A_0 id. v. – $\mathbf{H}^3$ □

$\mathcal{D}_{\mathbf{3}.1}$—$\mathbf{\Gamma_{3.1}}(3p;\ q,\ 2r;\ s=t;\ u)$ $1 \leq p$, $1 \leq r$, $2 \leq q, s, u$ – $\mathbf{\Gamma_{3.1}}$ is max. iff $q \neq 2r$ or $s \neq u$, else $\mathbf{\Gamma_1}(3p;\ q=2r;\ s=u)$ is supergroup – $\mathbf{\Gamma_{3.1}}$ is not optimal if $q=2$, then $\mathbf{\Gamma_{2.2}}(\bar{u}=p,\ \bar{v}=r,\ \bar{w}=s,\ \bar{x}=u)$ provides simpler presentation – $\mathbf{\Gamma_{3.1}}$ is not opt. if $p=1$, then $\mathbf{\Gamma_1}(\bar{u}=u,\ \bar{v}=r,\ \bar{w}=s,\ \bar{x}=q)$ is simpler – $\mathbf{\Gamma_{3.1}}$ is not opt. if $r=1$, then $\mathbf{\Gamma_1}(\bar{x}=p,\ \bar{y}=q,\ \bar{z}=s=t=u)$ is simpler • $m_0\colon {}^2A_1A_2A_3$, $m_1 : A_2A_3{}^1A_0$, $m_2 : A_3{}^1A_0{}^2A_0$, $m_2'\colon A_3{}^2A_0{}^2A_1$, $m_3\colon {}^1A_0{}^2A_0{}^2A_1A_2$ – m_0^2, m_1^2, m_2^2, ${m_2'}^2$, m_3^2; $(m_0m_1)^p$, $(m_0m_2')^2$, $(m_0m_3)^2$ $(m_1m_2)^q$, $(m_1m_3)^2$, $(m_2m_3)^s$, $(m_2'm_3)^u$ • $\Gamma^3(A_3) = (+,0;[\ ];\{(p,q,r,2)\})$ ∘ ${}^1\Gamma^0({}^1A_0) = (+,0;[\ ];\{(q,2,s)\})$ ∘ ${}^2\Gamma^0({}^2A_0) = (+,0;[\ ];\{(r,t,u)\})$ ∘ ${}^1\Gamma^1({}^1A_1) = (+,0;[\ ];\{(s,t)\})$ $\Rightarrow s=t$ • opt. $\Rightarrow A_3$ out. v. □

$\mathcal{D}_{\mathbf{3.2}}$—$\mathbf{\Gamma_{3.2}}(2p,\ q;\ 3r;\ s,\ 2t)$ $1 \leq r$; $1 \leq p, t$; $2 \leq q, s$ – *Sd:* $p < t$ or $p = t$, $q \leq s$ can be assumed – $\mathbf{\Gamma_{3.2}}$ is max. if $2p \neq q$ or $s \neq 2t$, else $\mathbf{\Gamma_1}(2p=q,\ 3r;\ s=2t)$ is supergroup – $\mathbf{\Gamma_{3.2}}$ is not opt. if $p=1$, $s=2$, then $\mathbf{\Gamma_{2.2}}(\bar{u}=t;\ \bar{v}=r;\ \bar{w}=2,\ \bar{x}=q)$ provides simpler presentation – dually $t=1$, $q=2$ leads to $\mathbf{\Gamma_{2.2}}$ as simpler presentation – $\mathbf{\Gamma_{3.2}}$ is not opt. if $r=1=p$, then $\mathbf{\Gamma_1}(\bar{x}=q,\ \bar{y}=s,\ \bar{z}=t)$ provides simpler presentation – dually $r=1=t$ leads to $\mathbf{\Gamma_1}$ as simpler presentation – $\mathbf{\Gamma_{3.2}}$ is not opt. if $p=1=t$, then $\mathbf{\Gamma_1}(\bar{x}=q;\ \bar{y}=r;\ \bar{z}=s)$ leads to simpler presentation • $m_0\colon {}^1A_1{}^1A_2A_3$, $m_0'\colon {}^2A_1{}^2A_2A_3{}^1A_2$, $m_1\colon {}^2A_2A_3A_0$,

$m_2\colon A_3A_0{}^1A_1$, $m_3\colon A_0{}^1A_1{}^1A_2{}^2A_1$, $m_3'\colon A_0{}^2A_1{}^2A_2$ – m_0^2, ${m_0'}^2$, m_1^2, m_2^2, m_3^2, ${m_3'}^2$; $(m_0m_0')^p$, $(m_0'm_1)^q$, $(m_0m_2)^2$, $(m_0m_3)^2$, $(m_0'm_3)^2$, $(m_0'm_3')^2$, $(m_1m_2)^r$, $(m_1m_3')^2$, $(m_2m_3)^s$, $(m_3m_3')^t$ • $\Gamma^3(A_3) = (+,0;\,[\,];\,\{(r,2,p,q)\})$ – (r,p,q)– $\mathbf{S}^2$: $(1,1,2)$, $(1,2,q)$, $(1,3,3)$, $(1,3,4)$, $(1,3,5)$ – $\mathbf{E}^2$: $(1,3,6)$, $(1,4,4)$, $(2,2,2)$ – $\mathbf{H}^2$: else ∘ $\Gamma^0(A_0) = (+,0;\,[\,];\,\{(r,2,t,s)\})$ – (r,t,s)– $\mathbf{S}^2$: $(1,1,2)$, $(1,2,s)$, $(1,3,3)$, $(1,3,4)$, $(1,3,5)$ – $\mathbf{E}^2$: $(1,3,6)$, $(1,4,4)$, $(2,2,2)$ – $\mathbf{H}^2$: else • $(p;q;r;s;t)$– $\mathbf{S}^3$: not opt., e.g.: $(1;3;1;3;3)$ or $(1;3;3;3;1)$ – $\mathbf{E}^3$: not opt., e.g.: $(1;4;3;4;1)$ and $(1;4;1;3;4)$ lead to **221.Pm$\bar{\mathbf{3}}$m**, $(2;4;1;4;2)$ leads to **123.P4/mmm** – $\mathbf{E}^3$ opt.: $(2;3;1;6;2)$ **191.P6/mmm** – $\mathbf{S}^2{\times}\mathbf{R}$ opt.: $(2;2;1;s;2)$ $2 \le s$, $(2;3;1;3;2)$, $(2;3;1;4;2)$, $(2;3;1;5;2)$ – $\mathbf{H}^2{\times}\mathbf{R}$ opt.: $(2;q;1;s;2)$ $\frac{1}{q}+\frac{1}{s}<\frac{1}{2}$ – $\mathbf{S}^2$**-splittings:**$r = 1 < p$, $3 \le t$; $\frac{1}{q}+\frac{1}{s}>\frac{1}{2}$ – e.g.: $(3;3;1;3;3) \to \mathbf{S}^3\colon \mathbf{\Gamma_1}(3;3;3) + \mathbf{S}^3\colon \mathbf{\Gamma_1}(3;3;3)$; $(3;4;1;3;4) \to \mathbf{S}^3\colon \mathbf{\Gamma_1}(3;4;3) + \mathbf{E}^3\colon \mathbf{\Gamma_1}(4;3;4)$; $(3;4;1;3;5) \to \mathbf{S}^3\colon \mathbf{\Gamma_1}(3;4;3) + \mathbf{H}^3\colon \mathbf{\Gamma_1}(4;3;5)$; $(4;3;1;4;4) \to \mathbf{E}^3\colon \mathbf{\Gamma_1}(4;3;4) + \mathbf{H}^3\colon \mathbf{\Gamma_1}(3;4;4)$; $(3;5;1;3;5) \to \mathbf{H}^3\colon \mathbf{\Gamma_1}(3;5;3) + \mathbf{H}^3\colon \mathbf{\Gamma_1}(5;3;5)$; – $\mathbf{E}^2$**-splittings:** $r = 1 < p$, $3 \le t$, $\frac{1}{q}+\frac{1}{s}=\frac{1}{2}$ – e.g.: $(2;3;1;6;3) \to \mathbf{E}^3\colon \mathbf{\Gamma_1}(2;3;6) + \mathbf{H}^3\colon \mathbf{\Gamma_1}(3;6;3)$; $(3;3;1;6;3) \to \mathbf{H}^3\colon \mathbf{\Gamma_1}(3;3;6) + \mathbf{H}^3\colon \mathbf{\Gamma_1}(3;6;3)$; – $\mathbf{H}^3$: not opt., e.g.: $(1;5;1;3;5)$ and $(1;5;3;5;1)$ lead to $\mathbf{\Gamma_1}(5;3;5)$ – $\mathbf{H}^3$ opt.: $r = 1$; $3 \le t$, $\frac{1}{q}+\frac{1}{s}<\frac{1}{2}$ – e.g.: $(2;4;1;5;3)$ – $\mathbf{E}^2$**-splitting:** $(2;2;2;2;2) \to \mathbf{H}^2{\times}\mathbf{R} + \mathbf{H}^2{\times}\mathbf{R}$ □

$\mathcal{D}_{\mathbf{3.3}}$—$\mathbf{\Gamma_{3.3}}(3p;\ 3q;\ r,\ 2s)$ $1 \le p,q$; $1 \le s$; $2 \le r$ – $\mathbf{\Gamma_{3.3}}$ is max. iff $r \ne 2s$, else $\mathbf{\Gamma_1}(3p;\ 3q;\ r = 2s)$ is supergroup – $\mathbf{\Gamma_{3.3}}$ is not opt. if $p = 1 = q$, then $\mathbf{\Gamma_{2.9}}(\bar{u} = 2;\ \bar{v} = 2;\ \bar{w} = r;\ \bar{x} = s)$ provides simpler presentation • $m_0\colon {}^1A_1A_2A_3$, $m_1\colon A_2A_3A_0'$, $m_2\colon A_3A_0{}^1A_1$, $m_3\colon A_0{}^1A_1A_2A_0'$, $r\colon A_3A_0{}^2A_1 \to A_3A_0'{}^2A_1$ – m_0^2, m_1^2, m_2^2, m_3^2, r^2; $(m_0m_1)^p$, $(m_0m_2)^2$, $(m_0m_3)^3$, $(m_1rm_2r)^q$, $(m_1m_3)^2$, $(m_2m_3)^2$, $(m_3rm_3r)^s$ • $\Gamma^3(A_3) = (+,0;\,[2];\,\{(p,2,q)\})$ – (p,q)– $\mathbf{S}^2$: $(1,1)$ is not opt. – $\mathbf{E}^2$: $(1,2)$ – $\mathbf{H}^2$: else ∘ $\Gamma^0(A_0,A_0') = (+,0;\,[\,];\,\{(q,2,s,r)\})$ – (q,s,r)– $\mathbf{S}^2$: $(1,1,2)$ is not opt., $(1,2,r)$, $(1,3,3)$, $(1,3,4)$, $(1,3,5)$ – $\mathbf{E}^2$: $(1,3,6)$, $(1,4,4)$, $(2,2,2)$ – $\mathbf{H}^2$: else • $(p;q;r;s)$– $\mathbf{S}^3$: not opt., e.g.: $(1;1;r;2)$, $(1;1;3;3)$, $(1;1;3;4)$ – $\mathbf{E}^3$: not opt., $(1;1;4;3)$ **229.Im$\bar{\mathbf{3}}$m** / $(1;2;2;1)$ A_3 id. v. – $\mathbf{H}^2{\times}\mathbf{R}$ opt.: $(1;2;r;1)$ $3 \le r$, A_3 id. v. / $(2;1;2;s)$ $2 \le s$, A_3 id. v. – $\mathbf{H}^3$ not opt. – $\mathbf{H}^3$ opt.: $(2;1;r;2)$ $3 \le r$ ($r = 4$ is not max.), $(2;1;3;3)$, $(2;1;3;4)$, $(2;1;3;5)$, $(2;1;4;3)$, $(2;1;5;3)$ A_3 id. v. / $(1;2;2;2)$; $(2;1;3;6)$, $(2;1;4;4)$ A_3 and (A_0,A_0') are id. v. □

$\mathcal{D}_{\mathbf{3.4}}$—$\mathbf{\Gamma_{3.4}}$ is dual to $\mathcal{D}_{\mathbf{3.1}}$—$\mathbf{\Gamma_{3.1}}$ □

$\mathcal{D}_{\mathbf{3.5}}$—$\mathbf{\Gamma_{3.5}}$ is dual to $\mathcal{D}_{\mathbf{3.3}}$—$\mathbf{\Gamma_{3.3}}$ □

Figures

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