On two methods of obtaining criteria of unimodality for density functions

By P. MEDGYESSY (Budapest)

Dedicated to Professor A. Rapcsák

1

Necessary and sufficient — or sufficient — conditions (C) of that a function be a characteristic function are well known ([1], Ch. 4). Now let (U) be some condition of that a function constructed by some characteristic function be the characteristic function of a unimodal distribution function. Then combining (U) with (C) one obtains a condition of that a function be the characteristic function of a unimodal distribution function.

The classical type of a condition (U) is presented by A. JA. HINČIN's well-known unimodality theorem ([2], p. 160; [3]; [4] p. 155, 501) which states that a function $\Phi(t)$ is the characteristic function of a (0) unimodal distribution function $F(x)$ if and only if it can be represented in the form $\Phi(t)=\frac{1}{t}\int_0^t \chi(u)\,du$ where $\chi(u)$ is some characteristic function*). If $\Phi(t)$ is differentiable, this is equivalent to that $\Phi(t)$ is the characteristic function of a (0) unimodal distribution function if and only if $\frac{d}{dt}[t\Phi(t)]=\chi(t)$ is a characteristic function.

Combining Hinčin's condition with G. PÓLYA's sufficient condition ([1], p. 74) of that a real, continuous, even function $\Phi(t)$ be a characteristic function, R. G. LAHA obtained sufficient conditions of that a real, even function be the characteristic function of a symmetric and unimodal distribution function ([5]; [6], p. 311).

Now it is clear that any other condition of type (C), combined with Hinčin's condition will yield a new condition of unimodality. For instance let (C) be the condition given by the

Theorem of M. Mathias. *Let $\Phi(t)$ be a real, bounded, continuous, even function such that $\int_{-\infty}^{+\infty} |\Phi(t)|\,dt<\infty$. Let us introduce, for $n=0, 1, \dots$ and $p>0$ the functions*

$$C_{2n}(p)=(-1)^n \int_{-\infty}^{\infty} \Phi(pt)e^{-t^2/2}He_{2n}(t)\,dt$$

*) (0) unimodal means: unimodal with mode at $x=0$.

$\left(He_\nu(t)=(-1)^\nu e^{t^2/2}\frac{d^\nu}{dt^\nu}(e^{-t^2/2})\right.$ *is the* ν *th Hermite polynomial.*$\left.\right)$ *Then the necessary and sufficient conditions of that* $\Phi(t)$ *be the characteristic function of a symmetric density function are* 1) $\Phi(0)=1$; 2) $C_{2n}(p)\geqq 0$ ([7]; [1] p. 68).

Now let us take Hinčin's condition in that form which claims $\frac{d}{dt}[t\Phi(t)]$ being some characteristic function. Combining these two conditions we easily obtain, as a counterpart of M. Mathias' theorem,

Theorem 1. *Let* $\varphi(t)$ *be a real, bounded, continuous, even function such that* $\int_{-\infty}^{\infty}|\varphi(t)|\,dt<\infty$, $\varphi'(t)$ *exists and is continuous and* $\int_{-\infty}^{\infty}|t\varphi'(t)|\,dt<\infty$. *Let us introduce, for* $n=0, 1, \ldots$ *and* $p>0$ *the functions*

$$M_{2n+1}(p)=(-1)^n\int_{-\infty}^{\infty}\varphi(pt)te^{-t^2/2}He_{2n+1}(t)\,dt.$$

Then the necessary and sufficient conditions of that $\varphi(t)$ *be the characteristic function of a symmetric,* (0) *unimodal density function are* 1) $\varphi(0)=1$; 2) $M_{2n+1}(p)\geqq 0$.

Proof. The function $\frac{d}{dt}[t\varphi'(t)]=\chi(t)$ satisfies the conditions in M. Mathias' theorem, taking into account that to the functions $C_{2n}(p)$, the functions

$$(-1)^n\int_{-\infty}^{\infty}\chi(pt)e^{-t^2/2}He_{2n}(t)\,dt=(-1)^n\int_{-\infty}^{\infty}pt\,\varphi(pt)e^{-t^2/2}He_{2n+1}(t)\,dt$$

correspond and $p>0$.

2

A new type of condition (U) can be obtained by the following idea.

Supposing that, in Hinčin's theorem, the density function $f(x)$ of $F(x)$ is differentiable and that the relevant conditions are fulfilled, one gets that the fact that $\frac{d}{dt}[t\Phi(t)]$ is a characteristic function is equivalent to that

$$-xf'(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}[t\Phi(t)]e^{-ixt}\,dt$$

is a density function. This observation suggests establishing the following

Theorem 2. *Let* $s(x)$ *be some real function such that* $s(-x)=s(x)$, $s(x)<0$ *for* $x>0$, $\int_{-\infty}^{\infty}|s(x)|\,dx<\infty$. *A function* $\varphi(t)$ *satisfying the conditions*

1) $\varphi(0)=1$;

2) $\int_{-\infty}^{\infty}|t\varphi(t)|\,dt<\infty$;

3) *if* $\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \varphi(t) e^{-ixt}\, dt = f(x)$, *then* $\int\limits_{-\infty}^{\infty} |f(x)|\, dx < \infty$;

is the characteristic function of a (0) *unimodal density function being differentiable except possibly at* $x=0$, *if and only if, at the notations*

$$\int\limits_{-\infty}^{\infty} s(x) e^{itx}\, dx = \sigma(t), \tag{1}$$

$$\chi(t) = -\frac{i}{2\pi} \int\limits_{-\infty}^{\infty} \sigma(t-u) u\, \varphi(u)\, du,$$

$\frac{\chi(t)}{\chi(0)}$ *is the characteristic function of some density function.*

Proof. By virtue of the conditions,

$$-\frac{i}{2\pi} \int\limits_{-\infty}^{\infty} \sigma(t-u) u\, \varphi(u)\, du = \chi(t) = \int\limits_{-\infty}^{\infty} s(x) f'(x) e^{itx}\, dx$$

and, if $f(x)$ is a (0) unimodal density function, then $\frac{s(x)f'(x)}{\chi(0)}$ is a density function, i.e. $\frac{\chi(t)}{\chi(0)}$ is a characteristic function. Conversely, if $\frac{\chi(t)}{\chi(0)}$ is a characteristic function, then $\frac{s(x)f'(x)}{\chi(0)}$ is a density function, and $f'(x) < 0$ for $x > 0$, $f'(x) > 0$ for $x < 0$, further, by $\varphi(0) = 1$, $\int\limits_{-\infty}^{\infty} f(x) dx = 1$; that is $f(x)$ is a (0) unimodal density function with characteristic function $\varphi(t)$.

Remark 1. Hinčin's condition can be obtained from (1) by taking $s(x) = -x \frac{e^{-x^2/2\varepsilon^2}}{\sqrt{2\pi}\varepsilon}$ $(\varepsilon > 0)$ i.e. writing $\sigma(u) = -i\varepsilon^2 u\, e^{-\varepsilon^2 u^2/2}$ in (1), performing partial integration and, finally, calculating the limit when $\varepsilon \to 0$.

Remark 2. In the foregoing we can take also $f(x+h) - f(x)$, $(h > 0, x > 0)$ instead of $f'(x)$ in $s(x) f'(x)$. Then the existence of $\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \varphi(t) e^{-ixt}\, dt = f(x)$ will clearly suffice. The corresponding Fourier transforms and calculations in concrete cases will become, however, more complicated. Therefore we avoid the relevant investigations.

Now we give an example to illustrate the power of Theorem 2, namely, we take as condition of type (U) the condition of unimodality expressed by this theorem and combine it with some condition of type (C). Evidently the crucial point of any application is the choice of $s(x)$ and $\sigma(t)$, respectively.

Example. Let (C) be the condition in M. Mathias' theorem of that a real, bounded, continuous, even function be a characteristic function. Combining it with the condition in Theorem 2, we obtain as the necessary and sufficient condition of that a real, bounded, continuous, even function $\varphi(t)$ satisfying the integrability conditions in Theorem 2 be the characteristic function of a (0) unimodal density function, that

$$(2) \qquad (-1)^n \int_{-\infty}^{\infty} \frac{\chi(pt)}{\chi(0)} e^{-t^2/2} He_{2n}(t)\, dt =$$

$$= \frac{(-1)^{n+1} i}{2\pi\, \chi(0)} \int_{-\infty}^{\infty} u\, \varphi(u) \Big[\int_{-\infty}^{\infty} \sigma(pt-u)\, e^{-t^2/2} He_{2n}(t)\, dt \Big]\, du \geqq 0$$

for $n=0, 1, \ldots$ and any $p>0$.

To show the effect of the choice of $s(x)$, let us take here now

$$s(x) = -x \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad \text{i.e.} \quad \sigma(t) = -ite^{-t^2/2}.$$

Then

$$\int_{-\infty}^{\infty} \sigma(pt-u) e^{-t^2/2} He_{2n}(t)\, dt =$$

$$= \frac{2\pi \sqrt{2\pi}\, i\, p^{n-1}}{(p^2+1)^{n+1}} e^{-\frac{1}{2}\left(\frac{u}{\sqrt{p^2+1}}\right)^2} He_{2n+1}\left(\frac{u}{\sqrt{p^2+1}}\right)$$

and for 2) we (obtain

$$\frac{(-1)^n \sqrt{2\pi}\, p^{n-1}}{\chi(0)(p^2+1)^n} \int_{-\infty}^{\infty} \varphi(Pu) u\, e^{-u^2/2} He_{2n+1}(u)\, du$$

for $n=0, 1, \ldots$ and $p>0$, $P>1$ $\left(P=\sqrt{p^2+1}\right)$, a condition very similar to that occurring in Theorem 1; let us notice that in $\varphi(Pu)$ $P>1$ already suffices.

Remark. The combination of the condition of unimodality expressed by Theorem 2 with G. Pólya's condition can be constructed automatically and is, therefore, omitted.

References

[1] E. Lukács, Fonctions caractéristiques. Dunod, Paris, 1964.

[2] B. V. Gnedenko—A. N. Kolmogorov, Limit distributions for sums of independent random variables. Addison — Wesley Publishing Company, Inc., Cambridge, Mass., 1954.

[3] P. Medgyessy, On the interconnection between the representation theorems of characteristic functions of unimodal distribution functions and of convex characteristic functions. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **8** (A) (1963), 425—430.

[4] W. Feller, An introduction to probability theory and its applications. Vol. 2. John Wiley and Sons, Inc., New York, 1966.

[5] R. G. Laha, On a class of unimodal distributions. *Proc. Amer. Math. Soc.* **12** (1961), 181—184.

[6] E. Lukács, Recent developments in the theory of characteristic functions. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. Vol. 2. University of California Press, Berkeley and Los Angeles, 1961; pp. 307—335.

[7] M. Mathias, Über positive Fourier-Integrale. *Math. Z.* **16** (1923), 103—125.

(Received October 15, 1973.)