On logarithmic summability of conjugate series of Fourier series

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1. As in [1], we say that an infinite sequence $\{S_n\}$ is summable by the logarithmic method of summability or summable (L) to the sum s if, for x in the interval (0, 1).

$$\lim_{x\to 1-0} \frac{1}{|\log(1-x)|} \sum_{n=1}^{\infty} \frac{S_n}{n} x^n = s,$$

which is written simply as $S_n \to s(\text{L})$.

In a paper [2], HSIANG has applied this method of summability to Fourier series of $f(x)$ and he obtained a summability criterion for it. The object of this paper is to make further application of this method in the theory of conjugate series of Fourier series.

2. Let $f(x)$ be a periodic function with period 2π, which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x). \tag{2.1}$$

Then the conjugate series of (2.1) is.

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx). \tag{2.2}$$

Fixing Θ, we write through out

$$\varphi(t) = \tfrac{1}{2}\{f(\Theta+t)+f(\Theta-t)-2s\},$$

$$\psi(t) = \tfrac{1}{2}\{f(\Theta+t)-f(\Theta-t)\}.$$

3. HSIANG [2] proved the following theorem:
If

$$\int_0^t |\varphi(u)|\, du = o(t|\log t|), \quad (t \to +0), \tag{3.1}$$

$$\int_t^{\delta} (|\varphi(u)|/u)\, du = o(|\log t|), \tag{3.2}$$

as $t\to+0$ for any arbitrary $0<\delta<\pi$, then the Fourier series of $f(x)$ is summable (L) to s at Θ.

We prove the following theorem:

Theorem: *If*

$$\int_0^t (|\psi(u)|/u)\,du = o(1), \quad (t\to+0), \tag{3.3}$$

$$\int_t^\delta (|\psi(u)|/u)\,du = o(|\log t|), \tag{3.4}$$

as $t\to+0$ *for any arbitrary* $0<\delta<\frac{\pi}{2}$, *then the conjugate series of Fourier series of* $f(x)$ *is summable* (L) *to*

$$\frac{1}{\pi}\int_0^\pi \psi(t)\cot(t/2)\,dt$$

at $x=\Theta$, *provided the sum integral exists in the Cauchy's sense.*

4. PROOF OF THE THEOREM. Let

$$\tilde{S}_n(\Theta) = \sum_{\nu=1}^{n}(b_\nu\cos\nu\Theta - a_\nu\sin\nu\Theta)$$

be the n-th partial sum of (2.2). This partial sum can be represented as a definite integral. Using Euler—Fourier formulae for the coefficients a_ν and b_ν we have

$$\begin{aligned}\tilde{S}_n(\Theta) &= \frac{1}{\pi}\sum_{\nu=1}^{n}\left(\cos\nu\Theta\int_{-\pi}^{\pi} f(t)\sin\nu t\,dt - \sin\nu\Theta\int_{-\pi}^{\pi} f(t)\cos\nu t\,dt\right)\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\sum_{\nu=1}^{n}(\cos\nu\Theta\sin\nu t - \sin\nu\Theta\cos\nu t)\right)dt =\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\sum_{\nu=1}^{n}\sin\nu(t-\Theta)\right)dt. \quad \text{(see § 2.3, [4])}.\end{aligned} \tag{4.1}$$

By putting $t-\Theta = u$, and a little simplification will give us a more suitable form of it.

$$\begin{aligned}\tilde{S}_n(\Theta) &= \frac{1}{\pi}\int_{-\pi-\Theta}^{\pi-\Theta} f(\Theta+u)\left(\sum_{\nu=1}^{n}\sin\nu u\right)du,\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\Theta+u)\left(\sum_{\nu=1}^{n}\sin\nu u\right)du.\end{aligned}$$

Breaking up the range of the integral, we have.

$$\begin{aligned}\tilde{S}_n(\Theta) &= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(\Theta+u)\left(\sum_{\nu=1}^{n}\sin\nu u\right)du + \int_0^\pi f(\Theta+u)\left(\sum_{\nu=1}^{n}\sin\nu u\right)du\right]\\ &= \frac{1}{\pi}\left[-\int_0^\pi f(\Theta-u)\left(\sum_{\nu=1}^{n}\sin\nu u\right)du + \int_0^\pi f(\Theta+u)\left(\sum_{\nu=1}^{n}\sin\nu u\right)du\right].\end{aligned}$$

This is obtained by writing $u=-v$ in the range $(-\pi, 0)$ i.e. in the first integral. So we have

$$\tilde{S}_n(\Theta)=\frac{1}{\pi}\int_0^{\pi}\{f(\Theta+u)-f(\Theta-u)\}\left(\sum_{\nu=1}^{n}\sin \nu u\right)du.$$

$$(4.2)\qquad =\frac{2}{\pi}\int_0^{\pi}\psi(t)\left(\sum_{\nu=1}^{n}\sin \nu t\right)dt.$$

Since

$$\sum_{\nu=1}^{n}\sin \nu t=\frac{\cos\frac{1}{2}t-\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}$$

(for this sum the reader may refer § 1.12, [4]) we have

$$(4.3)\qquad \tilde{S}_n(\Theta)=\frac{1}{\pi}\int_0^{\pi}\psi(t)\cdot\frac{\cos\frac{1}{2}t-\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}.$$

Using (4.3), we have

$$\sum_{n=1}^{\infty}\tilde{S}_n(\Theta)\frac{x^n}{n}=\sum_{n=1}^{\infty}\frac{x^n}{n}\left(\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{\cos(t/2)-\cos\left(n+\frac{1}{2}\right)t}{\sin(t/2)}\,dt\right)=$$

$$=\sum_{n=1}^{\infty}\frac{x^n}{n}\left(\frac{1}{\pi}\int_0^{\pi}\psi(t)\cot(t/2)\,dt\right)-$$

$$-\sum_{n=1}^{\infty}\frac{x^n}{n}\left(\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin(t/2)}\,dt\right).$$

Let us put

$$\frac{1}{\pi}\int_0^{\pi}\psi(t)\cot(t/2)\,dt=s.$$

Then we have

$$\sum_{n=1}^{\infty}\{\tilde{S}_n(\Theta)-s\}\frac{x^n}{n}=\frac{1}{\pi}\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}\left(-\sum_{n=1}^{\infty}\frac{x^n}{n}\cos\left(n+\frac{1}{2}\right)t\right)dt,$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}K(t,x)\,dt,$$

where $K(t,x)=\left(-\sum_{n=1}^{\infty}\frac{x^n}{n}\cos\left(n+\frac{1}{2}\right)t\right)$.

In order to prove the theorem, we have to show that

$$(4.4)\qquad \int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}K(t,x)\,dt=o\left(|\log(1-x)|\right)\quad\text{as}\quad x\to 1-0.$$

Using the common method of summation which is known as "$c+is$ method" we

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get the sum of the series as follows: Let

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \cos\left(n+\frac{1}{2}\right) t = c \quad \text{(say)}.$$

Then we have

$$C = x\cos\frac{3}{2}t + \frac{x^2}{2}\cos\frac{5}{2}t + \ldots + \frac{x^n}{n}\cos\left(n+\frac{1}{2}\right)t + \ldots .$$

The corresponding auxiliary sine series would be

$$S = x\sin\frac{3}{2}t + \frac{x^2}{2}\sin\frac{5}{2}t + \ldots + \frac{x^n}{n}\sin\left(n+\frac{1}{2}\right)t + \ldots .$$

Then we have

$$\begin{aligned} C+iS &= x\left(\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right) + \frac{x^2}{2}\left(\cos\frac{5}{2}t + i\sin\frac{5}{2}t\right) + \ldots \\ &\ldots + \frac{x^n}{n}\left(\cos\left(n+\frac{1}{2}\right)t + i\sin\left(n+\frac{1}{2}\right)t\right) + \ldots = \\ &= xe^{i\frac{3}{2}t} + \frac{x^2}{2}e^{i\frac{5}{2}t} + \ldots + \frac{x^n}{n}e^{i\left(n+\frac{1}{2}\right)t} + \ldots = \\ &= e^{it/2}\left(xe^{it} + \frac{x^2}{2}e^{2it} + \ldots + \frac{x^n}{n}e^{nit} + \ldots\right) = \\ &= -e^{it/2}\log_e(1-xe^{it}) = -e^{it/2}\log_e(1-x\cos t - ix\sin t). \end{aligned}$$

We know that general value of logarithm of a complex quantity $(\alpha+\beta i)$ is given by the formula.

$$\mathrm{Log}_e(\alpha+\beta i) = \log_e(\alpha^2+\beta^2)^{\frac{1}{2}} + i(2n\pi + \operatorname{arctg}\beta/\alpha)$$

and its principal value

$$\log_e(\alpha+\beta i) = \log_e(\alpha^2+\beta^2)^{\frac{1}{2}} + i\operatorname{arctg}(\beta/\alpha).$$

so we have

$$\begin{aligned} C+iS &= -e^{it/2}\left[\log_e\{(1-x\cos t)^2 + (-x\sin t)^2\}^{\frac{1}{2}} + i\operatorname{arctg}\frac{-x\sin t}{1-x\cos t}\right] \\ &= -e^{it/2}\left[\log_e(1-2x\cos t + x^2)^{\frac{1}{2}} - i\operatorname{arctg}\frac{x\sin t}{1-x\cos t}\right] \\ &= -\left(\cos\frac{t}{2} + i\sin\frac{t}{2}\right)\left[\log_e(1-2x\cos t+x^2)^{\frac{1}{2}} - i\operatorname{arctg}\frac{x\sin t}{1-x\cos t}\right]. \end{aligned}$$

Comparing real and imaginary parts, we have

$$C = -\left[\frac{1}{2}\cos\frac{t}{2}\log_e(1-2x\cos t+x^2) + \sin\frac{t}{2}\operatorname{arctg}\frac{x\sin t}{1-x\cos t}\right].$$

Hence we have

$$K(t, x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \cos\left(n+\frac{1}{2}\right) t =$$

$$= \frac{1}{2}\cos(t/2)\log_e(1-2x\cos t + x^2) + \sin(t/2)\operatorname{arctg}\frac{x\sin t}{1-x\cos t}.$$

Hence

$$\int_0^{\pi} \frac{\psi(t)}{\sin(t/2)} k(t, x)\, dt =$$

$$= \int_0^{\pi} \frac{\psi(t)}{\sin(t/2)} \left\{\frac{1}{2}\cos(t/2)\log_e(1-2x\cos t+x^2)\right\} +$$

$$+ \int_0^{\pi} \psi(t)\operatorname{arctg}(x\sin t/1 - x\cos t)\, dt =$$

$$= I_1 + I_2 \quad \text{(say)}.$$

Since the period of $\operatorname{tg}\alpha$ is π, we can consider

$$|\operatorname{arctg}(x\sin t/1-x\cos t)| < \pi/2$$

uniformly for $0 \leqq x < 1$ and $0 \leqq t \leqq \pi$, we find that $I_2 = O(1) = o(|\log_e(1-x)|)$, as $x \to 1-0$. Now we proceed to show that under the given conditions $I_1 = o(|\log_e(1-x)|)$ as $x \to 1-0$.

We break up the integral I_1 as follows:

$$I_1 = \int_0^{1-x} + \int_{1-x}^{\delta} + \int_{\delta}^{\pi}$$

$$= I_{11} + I_{12} + I_{13}.$$

We have by (3.3) and (3.4)

$$I_{11} = \int_0^{1-x} \frac{\psi(t)}{\sin(t/2)} \left\{\frac{1}{2}\cos(t/2)\log_e(1-2x\cos t+x^2)\right\} dt =$$

$$= O\left(\int_0^{1-x} \frac{|\psi(t)|}{t} |\log_e(1-2x\cos t+x^2)|\, dt\right) =$$

$$= O\left(|\log_e(1-2x\cos(1-x)+x^2)| \int_0^{1-x} \frac{|\psi(t)|}{t}\, dt\right) =$$

$$= o(|\log_e(1-x)|), \quad \text{as} \quad x \to 1-0;$$

$$I_{12} = \int_{1-x}^{\delta} \frac{\psi(t)}{\sin(t/2)} \left\{\frac{1}{2}\cos(t/2)\log_e(1-2x\cos t+x^2)\right\} dt =$$

$$= O\left(\int_{1-x}^{\delta} \frac{|\psi(t)|}{|\sin(t/2)|} |\log_e(1-2x\cos t+x^2)|\, dt\right).$$

Since $\frac{\sin t}{t}$ is steadily decreasing for $0<t<\frac{\pi}{2}$, we have $\frac{1}{\sin(t/2)} \leqq \frac{\pi}{t}$, (page 7, [3]); and also $|\log_e(1-2x\cos t+x^2)|$ attains its maximum at the upper limit of the integral $t=\delta$. We have $|\log_e(1-2x\cos t+x^2)| = O(1)$, as $x\to 1-0$, for the integral under consideration. Therefore, the estimate of I_{12} would be

$$I_{12} = O\left(\int_{1-x}^{\delta} \frac{|\psi(t)|}{t}\,dt\right) =$$

$$= o\big(|\log_e(1-x)|\big), \quad \text{as} \quad x\to 1-0;$$

Finally, we have

$$I_{13} = O\left(\log_e(1-x)\int_{\delta}^{\pi} \psi(t)\,dt\right) =$$

$$= O(1) = o\big(|\log_e(1-x)|\big), \quad \text{as} \quad x\to 1-0.$$

This completes the proof of the theorem.

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References

[1] D. BORWEIN, A logarithmic method of summability, *J. London Math. Soc.*, **33** (1958), 212—220.
[2] F. C. HSIANG, Summability (L) of Fourier series, *Bull. Amer. Math. Soc.*, **67** (1961), 150—153.
[3] E. C. TITCHMARSH, The theory of functions, *Oxford* (1958).
[4] A. ZYGMUND, Trigonometric series, Dover publications, (1955).

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