Convergence theorems of quasi-Hermite—Fejér interpolation

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The notion of quasi-Hermite—Fejér interpolation has been defined by P. SZÁSZ [1], afterwards A. SHARMA [2], K. K. MATHUR and R. B. SAXENA [3] continued the investigation in this direction.

Let there be given a sequence of real points

$$-1 = x_0 < x_1 < \dots < x_n < x_{n+1} = +1 \quad n = 1, 2, \dots \tag{1}$$

and let denote the corresponding arbitrary real values

$$y_0, y_1, \dots, y_n, y_{n+1} \tag{2}$$

and

$$y_1', \dots, y_n'. \tag{3}$$

By the notations

$$\omega(x) = \prod_{k=1}^{n} (x - x_k) \tag{4}$$

$$v_k(x) = 1 + \left[\frac{2x_k}{1-x_k^2} - \frac{\omega''(x_k)}{\omega'(x_k)}\right](x - x_k), \tag{5}$$

the quasi-Hermite—Fejér interpolatory polynomials $S_n(x)$ can be written in the form:

$$\begin{aligned} S_n(x) = {} & y_0 \frac{1-x}{2} \frac{\omega^2(x)}{\omega^2(-1)} + \sum_{k=1}^{n} \frac{1-x^2}{1-x_k^2} \cdot v_k(x) \left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 \cdot y_k + \\ & + \frac{1+x}{2} \frac{\omega^2(x)}{\omega^2(+1)} y_{n+1} + \sum_{k=1}^{n} \frac{1-x^2}{1-x_k^2} (x - x_k) \left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 \cdot y_k'. \end{aligned} \tag{6}$$

The polynomials $S_n(x)$ have the following properties:

$$S_n(x_k) = y_k \quad k = 0, 1, \dots, n+1; \tag{7}$$

$$S_n'(x_k) = y_k' \quad k = 1, 2, \dots, n; \tag{8}$$

that is $S_n(x)$ coincides with the values (2) at the points (1), but we only prescribe its derivatives for the points $x_1, x_2, \dots, x_n$.

P. Szász has introduced the notion of the strongly quasinormal point system (1).
We say, that the point system (1) is strongly quasinormal, if

$$(9)\qquad v_k(x) \geqq 2\varrho > 0 \quad x\in[-1, +1] \quad k = 1, 2, \dots, n; \quad n = 1, 2, \dots$$

where ϱ is independent of x, k and n. Since $v_k(x_k) = 1$, it follows, that $\varrho \leqq 1/2$. By the uniqueness of our interpolatory process it is well known, that if $P_m(x)$ is a polynomial of degree at most m, then

$$(10)\qquad \begin{aligned} P_m(x) \equiv \frac{1-x}{2}\cdot\frac{\omega^2(x)}{\omega^2(-1)}P_m(-1) + \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}\cdot v_k(x)\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 P_m(x_k) + \\ +\frac{1+x}{2}\cdot\frac{\omega^2(x)}{\omega^2(+1)}P_m(+1) + \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}\cdot(x-x_k)\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2\cdot P_m'(x_k), \end{aligned}$$

for $2n+1 > m$.

Now take $P_0(x)\equiv 1$ and we obtain the fundamental identity:

$$(11)\qquad \frac{1-x}{2}\cdot\frac{\omega^2(x)}{\omega^2(-1)} - \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}v_k(x)\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 + \frac{1+x}{2}\frac{\omega^2(x)}{\omega^2(+1)} \equiv 1,$$

furthermore for strongly quasinormal point systems on account of (9) we have

$$(12)\qquad \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 \leqq \frac{1}{2\varrho} \quad x\in[-1, +1]$$

and if $|x-x_k| \leqq 2$ then

$$(13)\qquad \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}|x-x_k|\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 \leqq \frac{1}{\varrho} \quad x\in[-1, +1].$$

Let be $f(x)\in C_1[-1, +1]$ and let designate in this case the linear operator belonging to $f(x)$

$$(14)\qquad \begin{aligned} S_n(f)(x) = \frac{1-x}{2}\frac{\omega^2(x)}{\omega^2(-1)}f(-1) + \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}v_k(x)\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 f(x_k) + \\ +\frac{1+x}{2}\frac{\omega^2(x)}{\omega^2(+1)}f(+1) + \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}(x-x_k)\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2\cdot f'(x_k). \end{aligned}$$

Let denote $C_k[-1, +1]$ the linear space of real-valued k times continuously differentiable functions on $[-1, +1]$ normed by

$$\|f\|_{C_k} = \max_{0\leqq i\leqq k}\ \max_{x\in[-1, +1]}|f^{(i)}(x)| \quad f(x)\in C_k[-1, +1].$$

We prove the following theorem:

Theorem 1. *Let be $f(x)\in C_1[-1, +1]$. Then for strongly quasinormal point systems* (1)

$$(15)\qquad \lim_{n\to\infty}\|S_n(f)-f\|_{C_0} = 0.$$

PROOF. By means of Weierstrass Theorem for arbitrary $\varepsilon>0$, there exists a polynomial P such that

$$\|f-P\|_{C_1} \leqq \varepsilon.$$

In the case of polynomials (10) is equivalent to

$$\lim_{n\to\infty} \|S_n(P)-P\|_{C_1} = 0.$$

On the other hand in view of linearity of S_n and the inequalities (11), (13)

$$\|S_n(f)-f\|_{C_0} \leqq \|S_n(f)-S_n(P)\|_{C_0} + \|S_n(P)-P\|_{C_0} + \|P-f\|_{C_0} \leqq$$
$$\varepsilon\left(1+\frac{1}{\varrho}\right)+0+\varepsilon = \varepsilon\left(2+\frac{1}{\varrho}\right),$$

for sufficiently large n, and this completes the proof.

We now turn to the examination of problem of the continuous functions.

Let be $f(x)\in C_0[-1, +1]$, and suppose that the point system (3) is bounded, that is

$$(16) \qquad |y_k'| \leqq M \quad k = 1, 2, \dots, n; \quad n = 1, 2, \dots$$

where M is independent of k and n.

Define the polynomial $S_n(f, y')\,(x)$ belonging to $f(x)\in C_0[-1, +1]$ and point system (16)

$$(17) \qquad \begin{aligned} S_n(f, y')(x) &= \frac{1-x}{2}\frac{\omega^2(x)}{\omega^2(-1)}f(-1) + \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}v_k(x)\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 f(x_k) + \\ &+\frac{1+x}{2}\frac{\omega^2(x)}{\omega^2(+1)}f(+1) + \sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}(x-x_k)\left[\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right]^2 y_k'. \end{aligned}$$

P. SZÁSZ has proved the following theorem:

Let be $f(x)\in C_0[-1, +1]$ and the point system (1) *strongly quasinormal. Furthermore, suppose that*

$$(18) \qquad \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1-x^2}{1-x_k^2}\frac{\omega^2(x)}{\omega'^2(x_k)} = 0 \quad x\in[-1, +1],$$

and the convergence is uniform. Then

$$\lim_{n\to\infty}\|S_n(f, y')-f\|_{C_0} = 0,$$

for all bounded point systems of (16) *type.*

One can ask that the assertion of the theorem without the assumption (18) is true. We now answer to the problem just posed. We prove that for strongly quasinormal point system the relation (18) holds uniformly.

In the proof we shall choose the method, which G. GRÜNWALD [4] has used in the case of ordinary Hermite—Fejér interpolation.

Let be

$$f(x) = \begin{cases} 0 & -1 \leqq x \leqq a \\ (x-a)^{\varrho} & a \leqq x \leqq +1 \end{cases} \tag{19}$$

where $0 < \varrho \leqq 1/2$, as in (9).

Let there be given, further, a sequence of functions by

$$f_\nu(x) = \begin{cases} 0 & -1 \leqq x \leqq a \\ -\nu^2(x-a)^{\varrho+2} + 2\nu(x-a)^{\varrho+1} & a \leqq x \leqq a+1/\nu \quad \nu = 1, 2, \dots \\ (x-a)^{\varrho} & a+1/\nu \leqq x \leqq +1. \end{cases} \tag{20}$$

Obviously $f_\nu(x) \in C_1[-1, +1]$ and one may see easily, that

$$\|f-f_\nu\|_{C_0} < 3\left(\frac{1}{\nu}\right)^{\varrho} \tag{21}$$

thus

$$\lim_{\nu\to\infty} \|f-f_\nu\|_{C_0} = 0. \tag{22}$$

If $a \neq x_k$ $(k=1, 2, \dots, n)$ then the derivatives $f'(x_k)$ exist and we can write by notation (14)

$$S_n(f)(a) = \sum_{a<x_k} \frac{1-a^2}{1-x_k^2} v_k(a) \left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2 (x_k-a)^{\varrho} +$$

$$+\frac{1+a}{2}\frac{\omega^2(a)}{\omega^2(+1)}(1-a)^{\varrho} + \sum_{a<x_k} \frac{1-a^2}{1-x_k^2}(a-x_k)\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2 \cdot \varrho \cdot (x_k-a)^{\varrho-1} =$$

$$= \sum_{a<x_k} \frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2 [v_k(a)-\varrho](x_k-a)^{\varrho} + \frac{1+a}{2}\frac{\omega^2(a)}{\omega'^2(x_k)}(1-a)^{\varrho} \geqq$$

$$\geqq \varrho \sum_{a<x_k} \frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2 (x_k-a)^{\varrho} + \frac{1+a}{2}\frac{\omega^2(a)}{\omega^2(+1)}(1-a)^{\varrho} \geqq 0.$$

In the first place we show, that

$$\lim_{n\to\infty}\left(\sum_{a\leqq x_k} \frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2 (x_k-a)^{\varrho} + \frac{1+a}{2}\frac{\omega^2(a)}{\omega^2(+1)}(1-a)^{\varrho}\right) = 0. \tag{24}$$

If $a = x_k$ then all members of left hand of (24) are 0, and thus (24) holds directly, hence we suppose, that $a \neq x_k$ $(k=1, 2, \dots, n)$.

Then $S_n(f)(a)$ can be written as

$$S_n(f)(a) = S_n(f-f_\nu)(a) + S_n(f_\nu)(a).$$

Given an $\varepsilon > 0$, select a positive integer ν such that

$$3\left(\frac{1}{\nu}\right)^{\varrho} < \varepsilon,$$

then for this ν on account of (21) we have

$$\|f-f_\nu\|_{C_1} < \varepsilon.$$

Since $f_\nu(x)\in C_1[-1,+1]$ and $f_\nu(a)=0$, by means of Theorem 1

$$|S_n(f_\nu)(a)|<\varepsilon,$$

for sufficiently large n.

Therefore using the relations (11), (13) and inequality

$$|a-x_k|\leqq\frac{1}{\nu},$$

$$|S_n(f)(a)|\leqq|S_n(f-f_\nu)(a)|+|S_n(f_\nu)(a)|\leqq$$

$$\leqq\left(\sum_{k=1}^{n}\frac{1-a^2}{1-x_k^2}v_k(a)\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2+\frac{1+a}{2}\frac{\omega^2(a)}{\omega^2(+1)}\right)\|f-f_\nu\|_{C_0}+$$

$$+\left|\sum_{a<x_k\leqq a+1/\nu}\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2[f'(x_k)-f_\nu'(x_k)]\right|+\varepsilon\leqq$$

$$\leqq\varepsilon+\sum_{a<x_k\leqq a+1/\nu}\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2\left[\varrho\left(\frac{1}{\nu}\right)^{\varrho}+(\varrho+2)\left(\frac{1}{\nu}\right)^{2}+2(\varrho+1)\left(\frac{1}{\nu}\right)^{\varrho}\right]+\varepsilon\leqq$$

$$\leqq 2\varepsilon+\frac{4\varrho+4}{2\varrho}\left(\frac{1}{\nu}\right)^{\varrho}\leqq 2\varepsilon+\frac{3}{\varrho}\left(\frac{1}{\nu}\right)^{\varrho}<3\varepsilon.$$

This means that (24) holds.

Similarly we obtain the following result:

$$\lim_{n\to\infty}\left(\frac{1-a}{2}\frac{\omega^2(a)}{\omega^2(-1)}(a+1)^{\varrho}+\sum_{a\geqq x_k}\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2(a-x_k)^{\varrho}\right]=0. \tag{25}$$

According to (24) and (25) we have:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2|a-x_k|^{\varrho}=0, \tag{26}$$

that is for arbitrary fixed $\delta>0$

$$\lim_{n\to\infty}\sum_{|a-x_k|>\delta}\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2=0. \tag{27}$$

Consequently for arbitrary positive $\varepsilon>0$

$$\sum_{k=1}^{n}|a-x_k|\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2=\sum_{|a-x_k|\leqq 2\varepsilon\varrho}\cdots+\sum_{|a-x_k|>2\varepsilon\varrho}\cdots\leqq$$

$$\leqq 2\varepsilon\varrho\sum_{k=1}^{n}\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2+2\sum_{|a-x_k|>2\varepsilon\varrho}\frac{1-a^2}{1-x_k^2}\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2\leqq \tag{28}$$

$$\leqq 2\varepsilon\varrho\frac{1}{2\varrho}+2\varepsilon_n=\varepsilon+2\varepsilon_n$$

where $\varepsilon_n\to 0$, as $n\to\infty$.

From the result (28) it follows that

$$\sum_{k=1}^{n}\frac{1-a^2}{1-x_k^2}\frac{\omega^2(a)}{\omega'^2(x_k)}\sum_{k=1}^{n}\frac{1-a^2}{1-x_k^2}|a-x_k|^2\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2\leqq$$

$$\leqq 2\sum_{k=1}^{n}\frac{1-a^2}{1-x_k^2}|a-x_k|\left[\frac{\omega(a)}{\omega'(x_k)(a-x_k)}\right]^2$$

that is

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1-a^2}{1-x_k^2}\frac{\omega^2(a)}{\omega'^2(x_k)}=0.$$

Since $a\in[-1,+1]$ is arbitrary, we obtain that the limit relation (18) is uniform on $[-1,+1]$. Finally we have the following result: for strongly quasinormal point system (18) holds a priori. After these we can state the next sharpening of the SzÁsz Theorem:

Theorem 2. *Let be $f(x)\in C_0[-1,+1]$ and the point system* (1) *strongly quasinormal. Then by notation* (17)

$$\lim_{n\to\infty}\|S_n(f,y')-f\|_{C_0}=0,$$

or all bounded point systems of (16) *type.*

In the relations (24) and (25) we obtained two interesting results:

$$\lim_{n\to\infty}\frac{1+a}{2}\frac{\omega^2(a)}{\omega^2(+1)}=0\quad a\in[-1,+1]\tag{29}$$

$$\lim_{n\to\infty}\frac{1-a}{2}\frac{\omega^2(a)}{\omega^2(-1)}=0\quad a\in[-1,+1]\tag{30}$$

and the convergence is uniform in both cases on $[-1,+1]$, provided, that the nodes make up a strongly quasinormal sequence.

It is known, that the zeros of Jacobi-polynomials $J_n^{(\alpha,\beta)}(x)$ form a strongly quasinormal point system in the case $0\leqq\alpha<1$ and $0\leqq\beta<1$.

Let designate $P_n(x)$ the Legendre-polynomials standardized by $P_n(+1)=1$. Then in view of (29) and (30)

$$\lim_{n\to\infty}P_n^2(x)=0\quad x\in(-1,+1)$$

and the convergence is uniform on the interval $[-1+\delta, 1-\delta]$ where $0<\delta<1$. The last result is in accord with the estimation of Bernstein—Stieltjes:

$$P_n^2(x)<\frac{C}{n}\frac{1}{\sqrt{1-x^2}}\quad x\in[-1,+1].$$

References

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