An extension of the Hilbert basis theorem to semirings

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1. Introduction

H. E. STONE [3] and P. J. ALLEN and E. J. BRACKEN [2] have given analogues of the Hilbert Basis Theorem for Halfrings and Boolean semirings, respectively. In this paper, an exact analogue of the Hilbert Basis Theorem will be given for a class of semirings that can be decomposed as a union of Noetherian rings.

2. Fundamentals

There are many different definitions of a semiring appering in the literature. Throughout this paper, a semiring will be defined as follows:

Definition 1. A set R together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectivelly) will be called a *pre-semiring* provided:

(i) $x+y = y+x$ for each $x, y \in R$

and

(ii) multiplication distributives over addition both from the left and from the right.

Definition 2. A pre-semiring R will be called a *semiring* if $\exists 0 \in R$ such that $x+0 = x$ and $x0 = 0x = 0$ for each $x \in R$. The semiring R is said to have an identity if $\exists 1 \in R$ such that $x1 = 1x = x$ for each $x \in R$.

If R' denotes a pre-semiring and 0 is any element not in R', then $R = R' \cup \{0\}$ is a semiring where the binary operations on R' are extended to R as follows: $x+0 = x$ and $x0 = 0x = 0$ for each $x \in R$.

Definition 3. A subset I of a semiring R will be called an *ideal* if $a, b \in I$ and $r \in R$ implies $a+b \in I$, $ar \in I$ and $ra \in I$.

Definition 4. An ideal I in the semiring R will be called a *k-ideal* if the following condition is satisfied: if $a \in I$, $b \in R$ and $a+b \in I$, then $b \in I$.

A ring with identity is called Noetherian if it satisfied the ascending chain condition for ideals. A Noetherian semiring is defined in an analogous manner as follows:

Definition 5. A semiring R with identity will be called a *Noetherian* semiring if $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$ is an ascending chain of ideals in R implies there exists a positive integer M such that $I_n = I_M$ for $n \geqq M$.

3. N-semirings and faithful ideals

Let L denote a meet-semilattice containing a greatest element and let $\alpha \wedge \beta = $ G.L.B. $\{\alpha, \beta\}$ for $\alpha, \beta \in L$. Let $\{R_\alpha | \alpha \in L\}$ denote a collection of rings with identities (the identity of R_α will be denoted by 1_α) where $R_\alpha \cap R_\beta = \emptyset$ if $\alpha \neq \beta$, and suppose when $\alpha, \beta \in L$ where $\alpha \geqq \beta$ there is a given ring homomorphism $\varphi_{\alpha,\beta}: R_\alpha \to R_\beta$ where $1_\alpha \varphi_{\alpha,\beta} = 1_\beta$. Moreover, assume $\varphi_{\alpha,\alpha}$ is the identity function on R_α and $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. The notation $\bigcup_{\alpha \in L} R_\alpha$ will denote the associated system consisting of the semilattice, the collection of rings and the homomorphisms as well as the set theoritic union of the rings. The system $\bigcup_{\alpha \in L} R_\alpha$ will be called a *union of rings*.

It is clear that $a_\alpha + b_\beta = a_\alpha \varphi_{\alpha,\alpha\wedge\beta} + b_\beta \varphi_{\beta,\alpha\wedge\beta}$ and $a_\alpha b_\beta = a_\alpha \varphi_{\alpha,\alpha\wedge\beta} b_\beta \varphi_{\beta,\alpha\wedge\beta}$ define binary operations on $\bigcup_{\alpha \in L} R_\alpha$ where $a_\alpha \in R_\alpha$ and $b_\beta \in R_\beta$. Straight forward calculations show that both of the binary operations are associative, that addition is commutative, and at that multiplication distributes over addition from the left and from the right. Since L has a greatest element, say γ, the multiplicative identity 1_γ of R_γ can be shown to be an identity of the system $\bigcup_{\alpha \in L} R_\alpha$. One has now obtained the following:

Theorem 6. *Every union of rings is a pre-semiring with identity.*

If $L = \{\alpha\}$ is a singleton set, then $\bigcup_{\alpha \in L} R_\alpha = R_\alpha$ is a ring, and consequently a semiring. When L has more than one element, it can easily be shown that $\bigcup_{\alpha \in L} R$ does not have a zero element. However, from the remark following Definition 2, the pre-semiring $\bigcup_{\alpha \in L} R_\alpha$ can be extended to a semiring $(\bigcup_{\alpha \in L} R_\alpha) \cup \{0\}$ with an identity. It will be understood that the system described as a *union of rings* will contain an attached zero when L has more than one element, and the system will be denoted by $\bigcup_{\alpha \in L} R_\alpha$.

Definition 7. R is said to be an N-semiring if R is the union of a finite number of Noetherian rings.

The above discussion shows the class of all N-semirings is a large class containing every Noetherian ring. One also has the following:

Theorem 8. *If R is an N-semiring, then R is Noetherian.*

PROOF. Suppose $R = \bigcup_{j=1} R_j$ is the union of a finite number of Noetherian rings and $\vartheta_1 \subseteq \vartheta_2 \subseteq \ldots \subseteq \vartheta_n \subseteq \vartheta_{n+1} \subseteq \ldots$ is an ascending chain of ideals in R. Let $I_n^j = \vartheta_n \cap R_j$ for $n = 1, 2, 3, \ldots$ and $j = 1, 2, \ldots, s$. Since $\vartheta_n \cap R_j \subseteq \vartheta_{n+1} \cap R_j$, it can be shown that $\{I_n^j\}_{n=1}^\infty$ is an ascending chain of ideals in the ring R_j, for each $j = 1, 2, \ldots, s$. Since R_j is Noetherian, $\exists M_j$ such that $n > M_j$ implies $I_n^j = I_{M_j}^j$. Let $M = \max\{M_j | j = 1, 2, \ldots, s\}$. If $n > M$, then $\vartheta_n = I_n^1 \cup I_n^2 \cup \ldots \cup I_n^s \cup \{0\}$, and each $I_n^j = I_M^j$. Consequently, $\vartheta_n = I_M^1 \cup I_M^2 \cup \ldots \cup I_M^s \cup \{0\} = \vartheta_M$ and it follows that R is a Noetherian semiring.

Definition 9. Suppose $R = \bigcup_{\alpha \in L} R$ is a union of rings, and 0 denotes the zero in R. Let $R^* = \{0_\alpha | \alpha \in L\} \cup \{0\}$. The elements in R^* will be called *local zero's* in R.

Since ring homomorphisms map zero's to zero's, it is an easy matter to show R^* is a subsemiring of $R = \bigcup_{\alpha \in L} R_\alpha$. If $r_\alpha \in R_\alpha$ and $0_\beta \in R_\beta$, then $r_\alpha 0_\beta = r_\alpha \varphi_{\alpha, \alpha \wedge \beta} 0_\beta \varphi_{\beta, \alpha \wedge \beta} = 0_{\alpha \wedge \beta} \in R^*$. Likewise $0_\beta r_\alpha \in R^*$ and one has the following:

Theorem 10. *If the semiring R is the union of rings, then R^* is an ideal in R.*

Let R denote a semiring with an identity. If x is transcendental over R and commutes with every element in R, then $R[x] = \left\{ \sum_{i=0}^{n} a_i x^i | a_i \in R \right\}$ will denote the set of all polynomials with coefficients in R. $R[x]$ forms a semiring with an identity under the usual operations addition and multiplication of polynomials. For a more detailed discussion of $R[x]$, see ALLEN [1].

Definition 11. Suppose the semiring R is a union of rings. The k-ideal A in $R[x]$ will be called a *faithful ideal* when $R^*[x] \subseteq A$.

When R is a ring, there is only one local zero, namely the zero of R. In this case, every ideal in $R[x]$ is a k-ideal and contains $R^*[x] = \{0\}$. Consequently, a faithful ideal is a complete generalization of an ideal in a polynomial ring.

4. An extension of the Hilbert basis theorem

Let A be an ideal in $R[x]$ where R is a union of rings. Let $L_i(A) = \{a \in R | a$ is the i^{th} coefficient of some $f(x) = \sum_{j=0}^{n} a_j x^j \in A$ and any coefficient a_j of $f(x)$ is a local zero if $j > i\}$. When $a \in L_i(A)$ and $f(x) \in A$ having a for its i^{th} coefficient and local zero's for coefficients of all terms of degree greater than i, $f(x)$ will be called an *associate* of a. With the aid of this notation, one has the following:

Lemma 12. *$\{L_i(A)\}_{i=0}^{\infty}$ is an ascending chain of ideals in the semiring $R = \bigcup_{\alpha \in L} R_\alpha$.*

PROOF. Let $a, b \in L_i(A)$ with associate polynomials $f(x)$ and $g(x)$, respectively, and let $r \in R$. One can show that $f(x) + g(x)$, $rf(x)$ and $f(x)r$ are associate polynomials of $a+b$, ra and ar, respectively. Consequently, $a+b \in L_i(A)$, $ra \in L_i(A)$ and $ar \in L_i(A)$ and it follows that each $L_i(A)$ is an ideal in R. It is clear that $1_\alpha x \in R[x]$ where 1_α is the identity of the ring R_α containing the element a. Since $(1_\alpha x) f(x) \in A$ with local zero's as coefficients of all terms of degree greater than $i+1$ and having a for its $i+1^{\text{th}}$ coefficient, it follows that $a \in L_{i+1}(A)$.

Lemma 13. *Let A and B be ideals in $R[x]$ where $R = \bigcup_{\alpha \in L} R_\alpha$ is the union of rings. Moreover, suppose $A \subseteq B$ and A is faithful in $R[x]$. If $L_i(A) = L_i(B)$ for $i = 0, 1, 2, \ldots,$ then $A = B$.*

PROOF. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in B$. Since $a_n \in L_n(B) = L_n(A)$, there exists $g_0'(x) \in A$ where a_n is the n^{th} coefficient of $g_0'(x)$ and all coefficients of terms of degree greater than n are local zero's in R. Let $g_0(x) = (-1)g_0'(x) \in A \subseteqq B$ where -1 is the inverse of the identity of the ring containing the element a_n. It is clear that $f(x)+g_0(x) \in B$ with local zero's for coefficients of all terms of degree greater than $n-1$. The above process can be repeated and one obtains a polynomial $g_1(x) \in A \subseteqq B$ where $f(x)+ +g_0(x)+g_1(x) \in B$ and has local zero's for all coefficients of terms of degree greater than $n-2$. After repeating the process a finite number of times, one has polynomials $g_0(x), g_1(x), \ldots, g_n(x) \in A$ such that $f(x)+g_0(x)+g_1(x)+\ldots+g_n(x) \in R^*[x]$; i.e., has only local zero's for its coefficients. Since A is faithful, $f(x)+g_0(x)+g_1(x)+\ldots+ \ldots+g_n(x) \in A$. Since A is a k-ideal and $g_0(x)+g_1(x)+\ldots+g_n(x) \in A$, it follows that $f(x) \in A$ and the proof is complete.

One can now obtain the following generalization of the Hilbert Basis Theorem:

Theorem 14. *If R is an N-semiring, then every ascending chain of faithful ideals in $R[x]$ is finite.*

PROOF. Let $I_0 \subseteqq I_1 \subseteqq I_2 \subseteqq \ldots \subseteqq I_n \subseteqq I_{n+1} \subseteqq \ldots$ be an ascending chain of faithful ideals in $R[x]$. The double sequence $\{L_i(I_n)\}$ gives rise to an ascending chain of coefficient ideals in R for either i or n fixed. Since the diagonal $\{L_q(I_q)\}_{q=0}^{\infty}$ is an ascending chain of ideals in R, there exists a positive integer K such that $L_q(I_q) = = L_K(I_K)$ when $q \geqq K$. Consequently, it can be shown that $L_K(I_K) = L_p(I_q)$ when $p \geqq K$ and $q \geqq K$. For each ascending chain $\{L_i(I_j)\}_{j=0}^{\infty}$, there exists a positive integer N_i such that $L_i(I_j) = L_i(I_{N_i})$ when $j \geqq N_i$. Letting $N = \max\{N_1, N_2, \ldots, N_{K-1}, K\}$, it is clear that $L_i(I_n) = L_i(I_N)$ for $n \geqq N$ and $i = 0, 1, 2, \ldots$. In view of the above lemma, $I_n = I_N$ for $n \geqq N$ and the proof is complete.

References

[1] PAUL J. ALLEN, Ideal Theory in Semirings, *Dissertation, Texas Christian University*, Fort Worth Texas, 1967.

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