On finiteness conditions for near rings

By HOWARD BELL and STEVE LIGH (Lafayette, La.)

1. Introduction

Finiteness conditions for various algebraic systems have been considered by several authors (see [1], [2], [7], [8], [9]). A very deep problem in group theory which remains unsolved is the following: Is a group finite, if it satisfies both the ascending and descending chain condition for subgroups? The answer is affirmative in the case of abelian groups. It was shown in [9] that a ring is finite if it satisfies both chain conditions for subrings. The purpose of this paper is to investigate the corresponding problem for near rings. While we do not have the general problem solved, we do have satisfactory answers for certain classes of near rings.

2. Preliminaries

A *near ring* R is a system with two binary operations, $+$ and $\cdot$ such that:

(i) $(R, +)$ is a group, not necessarily abelian,

(ii) $(R, \cdot)$ is a semigroup,

(iii) $x(y+z) = xy+xz$, for all $x, y, z \in R$,

(iv) $0x=0$ for all x in R.

In particular, if R contains a multiplicative semigroup S whose elements generate $(R, +)$ and satisfy

(v) $(x+y)s = xs+ys$, for all $x, y \in R$ and $s \in S$, we say that R is a distributively generated (d. g.) near ring.

The most natural example of a near ring is given by the set R of identity preserving mappings of an additive group G (not necessarily abelian) into itself. If the mappings are added by adding images and multiplication is composition of functions, then the system $(R, +, \cdot)$ is a near ring. If S is a multiplicative semigroup of endomorphisms of G and R' is the subnear-ring generated by S, then R' is a d.g. near ring. Other examples of d.g. near rings may be found in [6].

An element r of R is right (anti-right) distributive if $(b+c)r = br+cr$ $((b+c)r = cr+br)$, for all $b, c \in R$. It follows at once that an element r is right distributive if and only if $(-r)$ is anti-right distributive. In particular any element of a d.g. near ring is a finite sum of right and anti-right distributive elements.

A subgroup H of a near ring R is called an R-subgroup if $HR = \{hr: h\in H, r\in R\} \subseteq H$.

The kernels of near ring homomorphisms are called ideals. Equivalently, K is an ideal of a near ring R if and only if K is a normal subgroup of $(R, +)$ and satisfies

(i) $RK \subseteq K$ and

(ii) $(m+k)n-mn\in K$, for all $m, n\in R$ and $k\in K$.

For each x in a near ring R, the left annihilator $A(x)$ of x is defined to be the set $A(x) = \{r\in R: xr = 0\}$. In general, $A(x)$ is not an ideal of R.

3. Main results

It was shown in [9] (also [1], [8]) that if a ring R has a finite number of subrings, then R is finite. We now consider this problem for near rings and show that the result is valid for a certain class of near rings. But first a lemma is needed. The proof is trivial, hence omitted.

Lemma 3.1. *Let R be a near ring with no nonzero nilpotent elements. Then $xa=0$ implies that $ax=0$ and $xra=0$ for each r in R.*

Theorem 3.2. *Let R be a* d.g. *near ring with no nonzero nilpotent elements. If R has finitely many subnear-rings, then R is a finite commutative ring.*

PROOF. We use induction on $n(R)$, the number of proper subnear-rings of R.

(i) $n(R)=0$. For each $a\neq 0$ in R, aR is a subnear-ring of R and hence $aR=R$. By Theorem 3.4 in [4], R is a division ring. Hence R is a finite commutative ring.

(ii) $n(R)=k$. We consider two cases.

Case 1. R has no zero divisors. Consider the chain $aR\supseteq a^2R\supseteq \dots$. The fact that R has a finite number of subnear-rings and no zero divisors implies that $aR=R$ for each $a\neq 0$ in R. Again R is a division ring and result follows.

Case 2. R has zero divisors. For each zero divisor x in R, the set $A(x)$ is an ideal by straight forward calculation. It is clear that $0\neq A(x)\neq R$, and $R/A(x)$ has no nonzero nilpotent elements. Since a homomorphic image of a d.g. near ring is again d.g., we see that $R/A(x)$ satisfies the inductive hypothesis. Hence $R/A(x)$ is commutative. Thus for each $a, b\in R$, $(ab-ba)\in A(x)$. But the intersection of all $A(x)$, where x is a zero divisor, is zero. Hence R is a commutative (distributive) near ring. By Corollary 3 in [5], R is a ring. Hence R is a finite commutative ring. The proof is complete.

Before the next result we need a definition.

Definition 3.3. Let R be a d.g. near ring. The left annihilator L of R is defined by

$$L = \{x\in R: xr = 0 \quad \text{for all} \quad r\in R\}.$$

It can be verified easily that L is an ideal of R.

Theorem 3.4. *Let R be a* d.g. *near ring with the property that if x is nilpotent, then $xy=0$ for all y in R. If R has a finite number of subnear-rings, then R is finite.*

PROOF. If R has no nonzero nilpotent elements, then result follows from Theorem 3.2.

Now suppose R has nonzero nilpotent elements. Hence the left annihilator L of R is nontrivial. The near ring R/L is d.g. and has a finite number of subnear-rings. Furthermore it has no nonzero nilpotent elements. Thus by Theorem 3.2, R/L is finite. Observe that $ab=0$ for each $a, b \in L$. Thus every subgroup of L is a subnear-ring and since L can have only a finite number, it follows that L is finite. Now the fact that both L and R/L are finite implies that R is finite. The proof is complete.

It was shown in [6] that all the d.g. near rings defined on the symmetric groups S_n, $n \geqq 5$, satisfy the hypotheses of Theorem 3.4.

Now we consider the case when d.g. is removed from the hypothesis. We begin with the following result.

Proposition 3.5. *Let R be a near ring with a nonzero right distributive element d. If R has no proper subnearrings, then R is either the zero ring on C_p or the finite field Z_p for certain prime p.*

PROOF. If there exists an $x \neq 0$ in R sucht that $A(x) \neq 0$, then $A(x)=R$. Let $L(d) = \{y \in R \colon yd = 0\}$. Since d is right distributive, $L(d)$ is a subnear-ring and since $x \in L(d)$ we see that $L(d)=R$. It follows that R is the zero ring on the additive group generated by d, which is clearly isomorphic to C_p.

On the other hand, if R has no zero divisors, then $aR=R$ for each $a \neq 0$ in R. Now there exists e in R such that $de=d$ and $d(ed-d) = 0$ implies that $ed = d$. Let x be in R. Then $(xe-x)d = 0$ yields that $xe=x$ and consequently e is a right identity for R. Hence we see that R is a near field and $(R, +)$ is abelian. Now let

$$H = \{x \in R \colon x \text{ is right distributive}\}.$$

It can be verified easily that H is a subnear-ring of R. Since $e \in H$, it follows that $H=R$ and hence R is a division ring. But any division ring without a proper subring is the finite field Z_p.

Lemma 3.6. *Let R be a near ring with no nonzero nilpotent elements and at most one nonzero idempotent. If R has no proper subnear-rings, then R is a finite field.*

PROOF. For each $x \neq 0$ in R, $A(x)$ is a subnear-ring. Since $x^2 \neq 0$, we see that $A(x)=0$. Hence R has no zero divisors and $xR=R$. It can be easily verified that R is a near field and the set of right distributive elements form a subnear-ring. It follows that R is a finite division ring and hence a field.

Next we investigate the consequence of removing d.g. from the hypotheses of Theorem 3.2.

Theorem 3.7. *Let R be a near ring with no nonzero nilpotent elements. If R has at most one nonzero idempotent and finitely many subnear-rings, then R is a near field.*

PROOF. We use induction on $n(R)$, the number of proper subnear-rings of R.

(i) $n(R) = 0$. This is Lemma 3.6.

(ii) $n(R) = k$. If R has no zero divisors, then $aR = R$ for each $a \neq 0$ in R and R is a near field.

If R has a zero divisor $x \neq 0$, then $A(x) \neq 0$. Now by inductive hypothesis $A(x)$ is a near field. There is an element e in $A(x)$ such that $e^2 = e$. Also $xe = 0$ implies that $ex = 0$ and x is in $A(e)$. By induction $A(e)$ is a near field and hence contains a nonzero idempotent w. But $e \notin A(e)$ implies that $e \neq w$, a contradiction. This completes the proof.

Remark. It is not known whether or not the near field in Theorem 3.7. is finite.

4. Other finiteness conditions

Other finiteness conditions for rings and near reings involve finiteness of the number of zero divisors. It was shown in [2] that a ring with a finite number of nonzero divisors of zero is necessarily finite. This problem for near rings was studied in [7] and it was proved that a near ring R is finite if it has a finite number of nonzero right divisors of zero, while there is an infinite near ring with a finite number of nonzero left divisors of zero. However, it was shown that if one of the left zero divisor is right distributive, then the near ring is finite. Now we consider other condisions which along with a finite number of left zero divisors insure that a near ring is finite.

In the following "zero divisor" always means nonzero divisors of zero.

Theorem 4.1. *Let R be a* d.g. *near ring with* d.c.c. *on R-subgroups, and suppose moreover that R contains a nonzero idempotent which is not a left identity. If R has a finite number of left zero divisors, then R is finite.*

Proof. If the left annihilator L of R is trivial, then every right zero divisor x is also a left zero divisor. For if x is not a left zero divisor, then the chain $xR \supseteq x^2R \supseteq \ldots$ implies that $xR = R$. Suppose $a \neq 0$ and $ax = 0$. Let r be an arbitrary element of R. Then there exists a t in R such that $xt = r$. But

$$0 = ar = a(xt) = (ax)t.$$

This contradicts that L is trivial. This says that R can have only a finite number of right zero divisors. Hence by Theorem 2.1 in [7] R is finite.

Now suppose $0 \neq L \neq R$ and our result is not true. Let R be a minimal counterexample. By Theorem 2.3 in [7] we may assume that no nonzero right distributive element is a left zero divisor. Thus for each nonzero right distributive element d, $dR = d^2R = R$.

Fron hypotheses, suppose $e^2 = e$ and e is not a left identity. It is easily seen that $(e+L)$ is idempotent in R/L.

If $(e+L)$ is a left identity, then $(ed-d)$ is in L for each right distributive element d in R. Hence $(ed-d)d = 0$ implies that $ed^2 = d^2$. But $d^2R = R$ implies that if x is an arbitrary element in R, then there exists a y in R such that $x = d^2y$. Thus $ex = ed^2y = d^2y = x$. This is a contradiction.

Now R/L has d.c.c. and since $(e+L)$ is not a left identity, it follows that [3, Theorem 2] R/L has zero divisors. Let $(x+L)$ be a zero divisor in R/L. Then there exists $y \notin L$ such that $xy \in L$. Hence $(xy)r = 0$ for each r in R. Since $y \notin L$, there exists a t in R such that $yt \neq 0$. Hence $x(yt) = 0$ implies that x is a zero divisor of R. Thus

R/L has fewer left zero divisors than R. But this contradicts the minimality condition of R. Our theorem is now established.

T. SZELE has proved in [9] that if a ring R satisfies both the a.c.c. and d.c.c. on subrings, then R is finite. Now we consider a similar result for near rings.

Theorem 4.2. *Let R be a* d.g. *near ring with* d.c.c. *on subnear-rings. Then if R has a finite number of left zero divisors, R is locally finite with* a.c.c. *on R-subgroups.*

PROOF. First we establish local finiteness, that is, every finite set of elements generate a finite subnear-ring.

Case 1. R has no left zero divisors. Then R is a division ring, which must have characteristic p, for otherwise it would contain a copy of the ring of integers. If $\langle\ \rangle$ denotes "subring generated by", then for each a in R, $\langle a\rangle \supseteqq \langle a^2\rangle \supseteqq \langle a^4\rangle \supseteqq \dots$ must become stationary at some point and there exists a positive integer N such that $a^{2^N+1} \in \langle a^{2^N}\rangle$, so that a is algebraic over the prime subfield of R. It follows that $\langle a\rangle$ is finite and hence R has the "$a^n = a$" property and is therefore commutative. Local finiteness is immediate now.

Case 2. R has left zero divisors and the left annihilator L of R is trivial. Then, as in the first part of the proof of Theorem 4.1, R is finite.

Now suppose that $L \neq 0$ and our result is false. Let R be a minimal counterexample. If R/L has no zero divisors, then R/L is locally finite by Case 1 above. If R/L has zero divisors, then by the argument in the last part of the proof of Theorem 4.1, it has fewer left zero divisors than R, which is a contradiction. Hence, in either case, the finiteness of L plus local finiteness of R/L imply that R is locally finite.

Now we wish to establish that R satisfies a.c.c. on R-subgroups. From above R is locally finite. Hence for each x in R there exist different positive integers k and j such that $x^k = x^j$. Thus x is either nilpotent or some power of x is a nonzero idempotent.

Suppose that $A_1 \subseteqq A_2 \subseteqq \dots \subseteqq A_n \subseteqq \dots$ is an ascending chain of R-subgroups. If all the A_i consist of nilpotent elements, then the chain must become stationary because R has only a finite number of nilpotent elements. Otherwise some A_i contains a nonzero idempotent e. If e is not a left identity, R is finite by Theorem 4.1. If e is a left identity, then since A_i is an R-subgroup, $A_i = R$. This establishes that R satisfies a.c.c. on R-subgroups and the proof of our theorem is complete.

The following corollaries are immediate.

Corollary 4.3. Let R be a d.g. near ring having both the a.c.c. and d.c.c. on subnear-rings. If R has a finite number of left zero divisors, then R is finite.

Corollary 4.4. Let R be a d.g. near ring with a finite number of left zero divisors and a finite number of subnear-rings. Then R is finite.

Finally we would like to state the following two conjectures.

Conjecture 1. Let R be a d.g. near ring with a finite number of subnear-rings. Then R is finite.

Conjecture 2. There is an infinite near ring with a finite number of subnear-rings.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHWESTERN LOUISIANA
LAFAYETTE, LOUISIANA 70501
U.S.A.

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