A non-modular affine matroid lattice satisfying Euclid's strong parallel axiom is simple

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Dedicated to Professor András Rapcsák on his 60th birthday

§ 1. Introduction

O. ORE has proved in [5] that every partition lattice is simple (cf. [5, § 5 Theorem 8, p. 626]). Our aim is to prove the same theorem for another kind of matroid lattices. As a corollary we get a solution to a problem of M. F. JANOWITZ posed in [3]. Naturally, this solution follows as well from Ore's theorem.[1])

§ 2. Basic terminology

Let in this paragraph L be a lattice with 0. L is called weakly modular if $[a)$ is modular for every $a>0$ (cf. [4, Definition 1.10, p. 3]). We remark that G. GRÄTZER and E. T. SCHMIDT have defined weak modularity in quite another sense (cf. [1, Definition 4, p. 162]). In L perspectivity is introduced as follows (cf. [4, Definition 6.1, p. 26]): we say that $a, b \in L$ are perspective and write $a \sim_x b$ or simply $a \sim b$ when

$$a \cup x = b \cup x \quad \text{and} \quad a \cap x = b \cap x = 0$$

for some $x \in L$. L is called atomistic if every element of L is the join of a family of atoms. Let $F(L)$ denote the set of elements of L that may be expressed as a join of a finite (possibly empty) family of atoms. We say that in a lattice L the element $a \in L$ covers the element $b \in L$ and write $b \prec a$ in case $b < a$ and $b \leqq x \leqq a$ implies $x = b$ or $x = a$. The covering property (C) is introduced as follows: (C) if p is an atom and $p \nleqq a$ then $a \prec a \cup p$. We call L an AC-lattice if it is an atomistic lattice with covering property. In an AC-lattice the set $F(L)$ is always an ideal (cf. [4, Lemma 8.8, p. 37]). In every AC-lattice L one can define the height $h(a)$ of an element $a \in L$ (cf. [4, Definition 8.5, p. 36]). If L is an AC-lattice with 1 then $h(1)$ is called the length of L. An element $a \in L$ is called a line if $h(a) = 2$. A matroid lattice may be defined as an upper continuous AC-lattice.

Now following the terminology of G. GRÄTZER and E. T. SCHMIDT (cf. [2, p. 30]) we call an ideal S of a lattice L standard if $I \wedge (S \vee K) = (I \wedge S) \vee (I \wedge K)$ holds for

[1]) This was remarked by M. F. JANOWITZ in a written communication.

any pair of ideals I, K of L. An ideal S of L will be called homomorphism kernel if there exists a congruence relation Θ of L such that L/Θ has a zero element and $S=\{a\in L | a/\Theta=0/\Theta\}$. An ideal S of L is called p-ideal if L has a zero element and S is closed under perspectivity. The assertion of the following lemma is well-known, therefore we give no proof here.

Lemma 1. *Let L be a lattice with* 0 *and let S be an ideal of L. Then the following implications hold in L: S standard ideal $\Rightarrow S$ homomorphism kernel $\Rightarrow S$ p-ideal.*

§ 3. Affine matroid lattices

To introduce this kind of matroid lattices we need the notion of parallel elements

Definition 2 (cf. [4, p. 72]). Let L be a lattice with 0 and let $a, b\in L$ ($a\neq 0$, $b\neq 0$). We write $a<|b$ when

$$a\cap b=0 \quad \text{and} \quad b\prec a\cup b.$$

When $a<|b$ and $b<|a$ then we say that $a\in L$ and $b\in L$ are parallel and write $a\|b$.

Now we are ready to formulate Euclid's parallel axioms (cf. [4, Axiom 18.1 and Axiom 18.2, p. 78]).

Axiom 3 (Euclid's strong parallel axiom). *Let d be a line in a matroid lattice. If p is an atom with $p\not\leqq d$ then there exists one and only one line e such that $d\|e$ and $p<e$.*

Axiom 4 (Euclid's weak parallel axiom). *Let d be a line in a matroid lattice. If p is an atom with $p\not\leqq d$, then there exists at most one line e such that $d\|e$ and $p<e$.*

After this we can define affine matroid lattices.

Definition 5. (cf. [4, p. 78]). Let L be a weakly modular matroid lattice of length $\geqq 4$ (L may be of infinite length). When L satisfies Euclid's weak parallel axiom we call L an affine matroid lattice.

With the aid of parallelity we define in affine matroid lattices (more generally: in weakly modular AC-lattices) complete and incomplete elements.

Definition 6. (cf. [4, p. 78]). In an affine matroid lattice L a line d is called complete when there exists no line parallel to d, and d is called incomplete when there exists a line parallel to d. An element $a\in L$ of height $\geqq 2$ is called incomplete when every line contained in $a\in L$ is incomplete.

The following lemma tells us something more about the connection between incomplete elements and parallel elements.

Lemma 7. (cf. [4, Corollary 18.13, p. 81]). *Let L be an affine matroid lattice and let $a\in L$ be an element of height $\geqq 2$. Then the following statements are equivalent:*

(i) *$a\in L$ is incomplete and $a\neq 1$;*

(ii) *there exists an element $b\in L$ with $a\|b$.*

Finally we need a fact about non-modular affine matroid lattices.

Theorem 8. (cf. [4, Theorem 14.7, p. 61]). *A non-modular affine matroid lattice is irreducible and hence any two of its atoms are perspective.*

§ 4. Proof of the theorem

First we need the following

Lemma 9. *Let L be an affine matroid lattice, which satisfies Euclid's strong parallel axiom. Then there exists an atom $s \in L$ and a dual atom $n \in L$ such that $s \sim n$.*

Proof. We show that there are in L two dual atoms m_ν, n such that

$$(*) \qquad m_\nu \| n.$$

From this follows the assertion: take an atom $s < n$. Then $s \sim_{m_\nu} n$.

Now we prove $(*)$. Let

$$1 = \bigcup (a_\nu | \nu \in \Gamma)$$

where $\langle ..., a_\nu, ... \rangle_{\nu \in \Gamma}$ is a maximal independent set of atoms. We define

$$m_\nu \overset{\text{def}}{=} \bigcup (a_\mu | \mu \in \Gamma,\ \mu \neq \nu).$$

Then $m_\nu \cup a_\nu = 1$ and $m_\nu \cap a_\nu = 0$, therefore $a_\nu \not\leqq m_\nu$ and $m_\nu \not\leqq 1$. From $m_\nu < y \leqq \leqq m_\nu \cup a_\nu = 1$ we get by the covering property (C) that $y = 1$. Therefore m_ν is a dual atom.

Since in L every line is incomplete, every in m_ν contained line is incomplete. This means that m_ν is an incomplete element of L. Then m_ν fulfils condition (i) of Lemma 7. According to condition (ii) of the same lemma there exists an $n \in L$ such that $m_\nu \| n$. By definition 2 we have $n \prec m_\nu \cup n = 1$. Hence n is a dual atom which proves the lemma.

Now we can prove the theorem indicated in the title of this note.

Theorem 10. *Let L be a non-modular affin matroid lattice which satisfies Euclid's strong parallel axiom. Then L is a simple lattice.*

Proof. Let Θ be a congruence relation of L and suppose that

$$(1) \qquad \Theta > \omega$$

where ω is the least congruence relation of L. Consider

$$I = \{a \in L | a \equiv 0(\Theta)\}$$

I is the kernel of Θ and hence a homomorphism kernel. It follows from Lemma 1 that I is a p-ideal. Because of (1) there is an $(0 \neq) a \in L$ such that $a \in I$. L is atomistic, hence we can find an atom $q \in L$ such that $q \leqq a$. From this we get

$$(2) \qquad q \in I.$$

Let now $r (\neq q)$ be an arbitrary atom of L. Since L is non-modular, it is irreducible and we have

$$(3) \qquad q \sim r \quad \text{(cf. Theorem 8).}$$

Since I is a p-ideal, it follows from (2) and (3) that $r \in I$. This means that every atom of L is contained in I. Hence all finite unions of atoms are in I and therefore

$$(4) \qquad F(L) \subseteqq I.$$

Now by Lemma 9 there is an atom $s \in L$ and a dual atom $n \in L$ such that

(5) $$s \sim n.$$

Since I is a p-ideal, it follows from (4) and (5) that

(6) $$n \in I.$$

Since L is atomistic and n is a dual atom, there is an atom $t \in L$ such that

(7) $$t \not\leqq n.$$

By (4) we get

(8) $$t \in I.$$

Now (6), (7) and (8) give

$$1 = t \cup n \in I.$$

This means that $I = L$. Hence L has only the trivial congruence relations which proves the theorem.

Corollary 11. Let L be a non-modular affine matroid lattice which satisfies Euclid's strong parallel axiom. Then the following two conditions are equivalent:

(i) $F(L)$ is standard ideal in L;

(ii) L is of finite length.

PROOF. If L is of finite length then $F(L) = L$ is trivially a standard ideal of L. Let now $F(L)$ be a standard ideal of L. By Lemma 1 $F(L)$ is a homomorphism kernel of L. By Theorem 10 L is simple and therefore we get $F(L) = (0)$ or $F(L) = L$. In both cases L must be of finite length which was to be proved.

Corollary 12 (cf. [6, Theorem 6]). There exists a matroid lattice L such that $F(L)$ is not a standard ideal of L.

PROOF. Take a non-modular affin matroid lattice L of infinite length which satisfies Euclid's strong parallel axiom. Then by Corollary 11 $F(L)$ is not standard in L.

This corollary provides a solution to the following problem of M. F. JANOWITZ (cf. [3, Problem 4, p. 346]): Is $F(L)$ a standard ideal for L an arbitrary AC-lattice? What is if L is a matroid lattice?

References

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