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Translations in normed spaces

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Abstract. Using the methods of theory of functional equations we give some characterization of traslations in a real normed spaces.

S. Mazur and S. Ulam [2] have shown that every isometry of one real normed space X onto another Y is affine (i.e. $X \ni x \to f(x) - f(0) \in Y$ is linear). In [1] J. A. Baker has observed that the assumption "onto" is superflous in the case where Y is strictly convex. In this note we shall give a characterization of translation in the case of X being an arbitrary real linear normed space. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of all positive integers and the set of all reals, respectively, and for any $f : X \to X$ and $x \in X$ we put

$$f^0(x) := x \quad \text{and} \quad f^n(x) := f(f^{n-1}(x)), \quad n \in \mathbb{N}.$$

Theorem 1. *Let X be a real linear normed space and let $f : X \to X$ be an isometry satisfying the following assumptions;*

$$(1) \qquad \begin{cases} \text{there exists an } n \in \mathbb{N} \text{ such that the function} \\ X \ni x \to f^n(x) \in X \text{ has no fixed point} \end{cases}$$

and

$$(2) \qquad \text{for every } x \in X \text{ the points } x, f(x) \text{ and } f^2(x) \text{ are collinear.}$$

Then there exists an $a \in X \setminus \{0\}$ such that $f(x) = x + a, x \in X$.

Proof. By virtue of (2) there exists a function $\varphi : X \to \mathbb{R}$ such that

$$(3) \qquad f^2(x) - f(x) = \varphi(x)[f(x) - x], \qquad x \in X.$$

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Since f is an isometry and on account of (1) φ is a continuous function and $|\varphi(x)| = 1$ for every $x \in X$. Thus either

$$\varphi(x) = 1, \qquad x \in X, \tag{4}$$

or

$$\varphi(x) = -1, \qquad x \in X. \tag{5}$$

Assume (4). Then (3) can be written in the form

$$f^2(x) - f(x) = f(x) - x, \quad x \in X,$$

which (using the method of induction) gives

$$f(x) - x = f^{k+1}(x) - f^k(x), \quad x \in X, \quad k \in \mathbb{N}.$$

Consequently,

$$n[f(x) - x] = \sum_{k=0}^{n-1} [f^{k+1}(x) - f^k(x)] = f^n(x) - x, \quad x \in X, \quad n \in \mathbb{N}.$$

Hence we get the following representation

$$f(x) - x = \lim_{n\to\infty} \frac{f^n(x)}{n}.$$

Note that for $x, y \in X$ we have

$$\|f(x) - x - [f(y) - y]\| = \lim_{n\to\infty} \left\| \frac{f^n(x) - f^n(y)}{n} \right\| = \lim_{n\to\infty} \frac{\|x - y\|}{n} = 0,$$

which implies that

$$f(x) - x = f(y) - y, \quad x, y \in X.$$

Setting $y = 0$ and denoting $a := f(0)$ we obtain

$$f(x) = x + a, \quad x \in X.$$

To end the proof it is enough to show that condition (5) cannot hold. Indeed, assume (5). Now (3) has the form

$$f^2(x) = x, \qquad x \in X. \tag{6}$$

In particular f transforms X onto X. By a result of Mazur and Ulam [2] there exists a linear isometry $g : X \to X$ such that

$$f(x) = g(x) + a, \quad x \in X, \tag{7}$$

where

$$a = f(0). \tag{8}$$

Hence and by (6) $f(a) = 0$ and using (7) we get

$$g(a) = -a. \tag{9}$$

Now, by virtue of (7), the linearity of g, and (9)

$$f\left(\frac{a}{2}\right) = g\left(\frac{a}{2}\right) + a = \frac{1}{2}g(a) + a = \frac{a}{2},$$

which means that $\frac{a}{2}$ is a fixed point of f and contradicts (1). This ends the proof of Theorem 1.

Remark. The assumption (1) is essential in the Theorem 1. The function $f(x) = -x,\ x \in X$, is not a translation and fulfils condition (2).

To see the essence of assumption (2) let us consider the function f defined by the formula

$$f((x,y)) := (x+1, -y), \quad x, y \in \mathbb{R}.$$

It is easily seen that f is an isometry of $\mathbb{R}^2 (= \mathbb{R} \times \mathbb{R})$ onto itself and the points $(x,y),\ f(x,y)$ and $f^2(x,y)$ are collinear if and only if $y = 0$. Evidently f is not a translation.

If the condition (1) is not assumed we have the following

Theorem 2. *Let X be a real linear normed space and let f be an isometry of X into X satisfying the condition (2). Then there exist a constant a and a linear isometry g such that $f(x) = g(x) + a,\ x \in X$. Moreover, $g^2(x) = x$ for every $x \in X$.*

Proof. As in the proof of Theorem 1 we obtain condition (3). Now we define the sets $S_0,\ S_+$ and S_- as follows:

$$S_0 = \{x \in X;\ f(x) = x\},$$
$$S_+ = \{x \in X;\ f^2(x) - f(x) = f(x) - x\},$$
$$S_- = \{x \in X;\ f^2(x) = x\}.$$

It is not hard to check that

$$S_+ \cap S_- = S_0, \qquad S_+ \cup S_- = X. \tag{10}$$

We shall show that

$$f(S_0) \subset S_0, \tag{11}$$

$$f(S_+ \setminus S_0) \subset S_+ \setminus S_0 \tag{12}$$

and

$$f(S_- \setminus S_0) \subset S_- \setminus S_0. \tag{13}$$

The inclusion (11) is a simple consequence of the definition of S_0. Take an arbitrary $x \in S_+ \setminus S_0$. Assume that $f(x) \in S_0$. Then $f^2(x) = f(x)$ and by the definition of S_+ $f(x) = x$, which means that $x \in S_0$, a contradiction. Now assume that $f(x) \in S_- \setminus S_0$. Then $f^3(x) = f(x)$ and hence $f^2(x) = x$. Consequently $x \in S_-$. This contradiction proves (12). Take an arbitrary $x \in S_- \setminus S_0$. Assume that $f(x) \in S_0$. Hence and by the definitions of S_- and S_0 we have $x = f^2(x) = f(x)$ which implies that $x \in S_0$, a contradiction. If $f(x) \in S_+ \setminus S_0$ then $f^3(x) - f^2(x) = f^2(x) - f(x)$ and since $x \in S_- \setminus S_0$ then $f^2(x) = x$ and therefore $f^3(x) = f(x)$. Thus $f(x) - x = x - f(x)$ and, consequently, $f(x) = x$, a contradiction. This proves (13).

We shall consider two cases:

$$S_+ \setminus S_0 \neq \emptyset \tag{14}$$

and

$$S_+ \setminus S_0 = \emptyset. \tag{15}$$

First assume (14). Similarly as in the proof of Theorem 1 we get

$$f(x) = x + a, \quad x \in S_+ \setminus S_0, \tag{16}$$

where $a(\in X \setminus \{0\})$ is a constant. In this case S_0 has to be the empty set. In fact, if $x_0 \in S_0$ then for $x \in S_+ \setminus S_0$

$$\|x - x_0\| = \|f^n(x) - f^n(x_0)\| = \|x + na - x_0\|, \; n \in \mathbb{N},$$

which can be written in the form

$$\frac{\|x - x_0\|}{n} = \left\| \frac{x}{n} + a - \frac{x_0}{n} \right\|.$$

Letting n tend to infinity we obtain $a = 0$, a contradiction. By the definition of $S_0, \ S_+, \ S_-$, and by (3) and (10) we get $S_+ \setminus S_0 = X$ and therefore on account of (16) we have $f(x) = x + a, \ x \in X$.

Now, assume (15). According to (10) and by the definition of S_-

$$f^2(x) = x, \quad x \in X. \tag{17}$$

In particular f transforms X onto X and by a result of Mazur and Ulam mentioned in the proof of Theorem 1 there exists a linear isometry $g : X \to X$ fulfilling the conditions (7) and (9). Moreover, by the linearity of g, (17) and (9) we obtain

$$\begin{aligned} g^2(x) &= g(g(x)) = g(f(x) - a) = g(f(x)) - g(a) \\ &= f(f(x)) - a - g(a) = f^2(x) = x. \end{aligned}$$

This ends the proof of Theorem 2.

The proof of Theorem 2 yields the following

Corollary. *If f fulfills the assumptions of Theorem 2 then either $f(x) = x + a$, $x \in X$, with some $a \in X \setminus \{0\}$ or f is an involution (i.e. $f^2(x) = x$, $x \in X$).*

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