The integrability class of the sine transform of a monotonic function

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1. A non-decreasing, continuous and real-valued function Φ defined on the non-negative half line and vanishing only at the origin is called an Orlicz function (OF). Function $\Phi \in OF$ is a said to satisfy Δ_2 condition for large u if there are constants $C>0$ and $u_0 \geqq 0$ such that $\Phi(2u) \leqq C\Phi(u)$, $u \geqq u_0$.

2. Recently BOAS [3] has proved the following theorem for Fourier transform by a method which is rather more direct than those that have been used for similar problems about Fourier series. His method depends on the Steffensen's version of Jensen's inequality (see MITRINOVIĆ [5], p. 109) and a theorem of EDMOND'S [4] on Parseval's theorem for monotonic functions.

Theorem A. *If* $f(x)\downarrow 0$, $x^{1/p}f(x) \in L^p(0, 1)$ *and*

$$F(x) = \sqrt{\left(\frac{\pi}{2}\right)} \int_0^\infty f(t) \sin xt \, dt,$$

then $x^{r+1-2/p}F(x) \in L^p(0, \infty)$ *provided that* $x^{-r}f(x) \in L^p(0, \infty)$, *where* $p>1$ *and* $-1/p<r<1/p$.

It may be remarked that in Theorem A, the condition $x^{1/p}f(x) \in L^p(0, 1)$ need not be mentioned because it is already implied by the condition $x^{-r}f(x) \in L^p(0, \infty)$, $-1/p<r<1/p$.

In this note it is proposed to obtain a generalization of the above theorem. Instead of considering L^p class we would employ a more general class, namely L_Φ.

3. We prove the following theorem:

Theorem. *Let* $F(x)$ *be the sine transform of* $f(x)$. *If* $f(x)\downarrow 0$, $x^{-\alpha}\Phi(f(x)) \in L(0, \infty)$ *and* $-1<\alpha<1$, *then* $x^{\alpha-2}\Phi(xF(x)) \in L(0, \infty)$, *where* $\Phi(x)$ *is a convex Orlicz function satisfying* Δ_2 *condition.*

It may be observed that for $\Phi(t)=t^p$, $p>1$, we get Theorem A.

4. We require the following lemmas for the proof of our theorems.

Lemma 1 ([2]). *Let* λ *be a function of bounded variation on every finite sub interval of* $(0, \infty)$; $\lambda(0) \leqq \lambda(x)$ *for all* $x>0$; *and* $\lambda(0) < \Lambda = \sup \lambda(x)$. *Let* $f(x)$ *decrease and*

$f(x) \geqq 0$. If ψ is continuous and convex over $(0, f(0))$, $\psi(0) \leqq 0$ and $\int_0^\infty d\mu(x) \geqq \Lambda - \lambda(0)$, then

$$\psi \left\{ \frac{\int_0^\infty f(x)\, d\lambda(x)}{\int_0^\infty d\mu(x)} \right\} \leqq \frac{\int_0^\infty \psi(f(x))\, d\lambda(x)}{\int_0^\infty d\mu(x)}.$$

Lemma 2 ([1], p. 58). *If g and B decrease to 0 on $(0, \infty)$ and $xg(x)$, $xB(x) \in L(0, 1)$, then $B(y)\, b(y) \in L(0, \infty)$ iff $g(u)G(u) \in L(0, \infty)$ and Parseval's formula*

$$\int_0^\infty B(y) b(y)\, dy = \int_0^\infty G(u) g(u)\, du$$

holds, G and b being the sine transforms of B and g respectively.

5. Proof of the Theorem. Taking $\lambda(x) = 1 - \cos x$, $\Lambda = 2$ and using Lemma 1, we have

$$\Phi\left(\frac{1}{2}\int_0^\infty f(x) \sin x\, dx\right) \leqq \frac{1}{2}\int_0^\infty \Phi(f(x)) \sin x\, dx. \tag{5.1}$$

Since sine transform of a positive decreasing function is positive, it follows that right-hand side is positive. Also in view of the hypothesis, it is finite. Now replacing $f(x)$ by $f(xt)$, multiplying (5.1) by $t^{-\alpha}$ and integrating over $(0, \infty)$ we have

$$\int_0^\infty t^{-\alpha} \Phi\left(\frac{1}{2}\int_0^\infty f(xt) \sin x\, dx\right) dt \leqq \frac{1}{2}\int_0^\infty t^{-\alpha}\, dt \int_0^\infty \Phi(f(xt)) \sin x\, dx. \tag{5.2}$$

Putting $t = 1/y$ and $x = yu$ in (5.2) we have

$$\int_0^\infty y^{\alpha-2} \Phi\left(\frac{1}{2}\int_0^\infty f(u) \sin yuy\, du\right) dy \leqq \frac{1}{2}\int_0^\infty y^{\alpha-2}\, dy \int_0^\infty \Phi(f(u)) \sin yuy\, du.$$

That is to say

$$\int_0^\infty y^{\alpha-2} \Phi\left(\frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}\, yF(y)\right) dy \leqq \frac{1}{2}\int_0^\infty y^{\alpha-1}\, dy \int_0^\infty \Phi(f(u)) \sin yu\, du.$$

Thus it follows that

$$\begin{aligned} \int_0^\infty y^{\alpha-2} \Phi(yF(y))\, dy &\leqq C \int_0^\infty y^{\alpha-2} \Phi\left(\frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}\, yF(y)\right) dy \\ &\leqq C \int_0^\infty y^{\alpha-1}\, dy \int_0^\infty \Phi(f(u)) \sin yu\, du \\ &= C \int_0^\infty B(y) b(y)\, dy, \end{aligned}$$

where $B(y)=y^{\alpha-1}$, $-1<\alpha<1$, $g(u)=\Phi(f(u))$ and G and b are the sine transforms of B and g respectively.

Now,

$$\int_0^\infty g(u)G(u)\,du = \int_0^\infty \Phi(f(u))\,du \int_0^\infty y^{\alpha-1}\sin yu\,du$$

$$= \int_0^\infty u^{-\alpha}\Phi(f(u))\,du \int_0^\infty t^{\alpha-1}\sin t\,dt\ ^{*)}$$

$$= \Gamma(\alpha)\sin\frac{\pi\alpha}{2}\int_0^\infty u^{-\alpha}\Phi(f(u))\,du$$

$$< \infty,$$

by virtue of the hypothesis. Thus $Gg\in L(0,\infty)$. In view of Lemma 2, Parseval's formula holds and therefore

$$\int_0^\infty y^{\alpha-2}\Phi(yF(y))\,dy<\infty.$$

Thus our theorem is proved.

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References

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*) When $\alpha=0$, the integral $\int_0^\infty \frac{\sin t}{t}\,dt=\pi/2$.