On conformally flat generalised 2-recurrent spaces

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Introduction. A non-flat Riemannian space whose curvature tensor R_{hijK} satisfies the relation

$$\nabla_m \nabla_l R_{hijK} = \beta_m \nabla_l R_{hijK} + a_{lm} R_{hijK} \tag{1}$$

where β_m and a_{lm} are not both zero and ∇ denotes the operator of covariant differentiation with respect to the metric of the space has been called elsewhere [1] a generalised 2-recurrent space and an n-space of this kind has been denoted by $G(^2K_n)$. If $\beta_m=0$, the space reduces to what is known to be a 2-recurrent space as named by A. LICHNEROWICZ [2]. Considering conformally flat 2-recurrent spaces ROY CHOWDHURY [3] showed that the scalar curvature of such a space is necessarily zero. Later, A. H. THOMPSON [4] proved that all n-dimensional ($n \geqq 3$) conformally flat 2-recurrent spaces with indefinite metric are recurrent and that every such space with definite metric is flat. In the present paper we consider conformally flat generalised 2-recurrent spaces with symmetric tensors of recurrence. In such a space of zero scalar curvature, canonical expressions have been given for the Ricci tensor and the recurrence tensor in terms of a real null vector field and a non-zero scalar field. The question whether such a space can have definite metric has been answered and the nature of the space in case of indefinite metric has been determined.

1. *Conformally flat* $G(^2K_n)$ ($n \geqq 3$) *with symmetric recurrence tensor.* Let us suppose that a $G(^2K_n)$ with symmetric tensor of recurrence a_{lm} is conformally flat. Then its curvature tensor is of the form

$$R_{hijK} = g_{hj} H_{iK} - g_{hK} H_{ij} + g_{iK} H_{hj} - g_{ij} H_{hK}, \tag{1.1}$$

where

$$H_{ij} = -\frac{1}{n-2}\left[R_{ij} - \frac{R}{2(n-1)} g_{ij}\right] \tag{1.2}$$

and further [5]

$$\nabla_K H_{ij} = \nabla_j H_{iK}. \tag{1.3}$$

In consequence of (1) it follows from (1.2) that

$$\nabla_l \nabla_K H_{ij} = \beta_l \nabla_K H_{ij} + a_{Kl} H_{ij}. \tag{1.4}$$

Hence (1.3) gives

$$a_{Kl} H_{ij} = a_{jl} H_{iK}. \tag{1.5}$$

Transvecting (1.5) with g^{Kl} we get

$$\Theta H_{ij} = a_{jl} g^{Kl} H_{iK},$$

where $\Theta = g^{Kl} a_{Kl}$, whence [1])

$$\Theta R_{ij} = \frac{R}{2}\frac{n-2}{n-1} a_{ij} + \frac{R}{2}\frac{\Theta}{n-1} g_{ij}. \tag{1.6}$$

If $\Theta = 0$ it follows from (1.6) that $R = 0$ because $n > 2$ and $a_{ij} \neq 0$.

If $\Theta \neq 0$, then $R \neq 0$ for $R = 0$ would in this case imply $R_{ij} = 0$ which makes the space flat, contrary to assumption. We can therefore state the following theorem.

Theorem 1. *In a conformally flat $G(^2K_n)$ with symmetric recurrence tensor. the scalar curvature and the scalar Θ are either both zero or both different from zero,*

2. *Conformally flat $G(^2K_n)$ with symmetric recurrence tensor and zero scalar curvature.*

In this case we suppose $R_{ij} \neq 0$ and $R = 0$. Hence there exists a real vector field λ^i such that

$$R_{ij} \lambda^i \lambda^j = e \quad (e = \pm 1).$$

Let $\mu_i \overset{\text{def}}{=} R_{ij} \lambda^j$. Then we have $\lambda^i \mu_i = e$.

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[1]) Proof of (1.6): From the Bianchi identity we have

$$\nabla_m \nabla_l R_{hijK} + \nabla_m \nabla_j R_{hiKl} + \nabla_m \nabla_K R_{hilj} = 0$$

$$\text{or,}\quad a_{lm} R_{hijK} + a_{jm} R_{hiKl} + a_{Km} R_{hilj} + \beta_m (\nabla_l R_{hijK} + \nabla_j R_{hiKl} + \nabla_K R_{hilj}) = 0$$

$$\text{or,}\quad a_{lm} R_{hijK} + a_{jm} R_{hiKl} + a_{Km} R_{hilj} = 0 \text{ (using Bianchi's identity)}$$

$$\text{or,}\quad a_{lm} R^t_{ijK} + a_{jm} R^t_{iKl} + a_{Km} R^t_{ilj} = 0.$$

Contracting t and l we get

$$a_{tm} R^t_{ijK} = a_{Km} R_{ij} - a_{jm} R_{iK}.$$

Transvecting with g^{ij} we get $a_{tm} R^t_K = a_{Km} R - a_{jm} R^j_K$ whence

$$a_{jm} R^j_K = \tfrac{1}{2} R a_{Km}. \tag{$*$}$$

Now from $\Theta H_{ij} = a_{jl} g^{Kl} H_{iK}$ we get using (1.2)

$$-\frac{1}{n-2}\Theta\left[R_{ij} - \frac{R}{2(n-1)} g_{ij}\right] = -\frac{1}{n-2} a_{jl} g^{Kl}\left[R_{iK} - \frac{R}{2(n-1)} g_{iK}\right]$$

or,

$$\Theta R_{ij} - \frac{R\Theta}{2(n-1)} g_{ij} = a_{jl} R^l_i - \frac{R}{2(n-1)} a_{ji} = a_{lj} R^l_i - \frac{R}{2(n-1)} a_{ij} \quad (a_{ij} \text{ is symmetric})$$

$$= \frac{1}{2} R a_{ij} - \frac{R}{2(n-1)} a_{ij} \quad (\text{by } (*))$$

$$= \frac{R}{2}\frac{n-2}{n-1} a_{ij}.$$

In consequence of (1.5) we have

$$R_{ij} a_{Kl} = R_{iK} a_{jl}.$$

So

$$R_{ij} a_{Kl} \lambda^K = \mu_i a_{jl}. \tag{2.1}$$

Contraction with λ^i gives

$$e a_{jl} = \mu_j a_{Kl} \lambda^K \tag{2.2}$$

Let

$$\varrho \overset{\text{def}}{=} a_{ij} \lambda^i \lambda^j.$$

Then

$$\mu_j \varrho = e a_{jl} \lambda^l,$$

i. e.:

$$e \mu_l = a_{Kl} \lambda^K.$$

Thus we have[2]) from (2.2)

$$a_{jl} = \varrho \mu_j \mu_l. \tag{2.3}$$

Since $a_{jl} \neq 0$, $\varrho \neq 0$. From (2.1) we get

$$R_{ij} \varrho \mu_K \mu_l \lambda^K = \mu_i a_{jl}.$$

Whence

$$R_{ij} = e \mu_i \mu_j. \tag{2.4}$$

Since $R=0$, it follows from (2.4) that μ_i is a null vector field.

If the metric is definite it cannot contain a real null vector field. Hence the assumption that $R_{ij} \neq 0$ is false. We can therefore state the following theorems:

Theorem 2. *In a conformally flat $G(^2K_n)$ with symmetric a_{lm} and zero scalar curvature, there exist a real null vector field μ_i and a non-zero scalar field ϱ such that the Ricci tensor R_{ij} and the recurrence tensor a_{ij} have the canonical forms:*

$$R_{ij} = e \mu_i \mu_j (e = \pm 1) \quad \text{and} \quad a_{ij} = \varrho \mu_i \mu_j.$$

Theorem 3. *A conformally flat $G(^2K_n)$ with symmetric recurrence tensor and zero scalar curvature cannot admit a definite metric.*

Since $\nabla_K R_{ij} = \nabla_j R_{iK}$ and $R_{ij} = e \mu_i \mu_j$,

we get

$$\mu_i (\nabla_K \mu_j - \nabla_j \mu_K) + (\nabla_K \mu_i) \mu_j - (\nabla_j \mu_i) \mu_K = 0. \tag{2.5}$$

Let $t_K = \lambda^i \nabla_K \mu_i$. Then transvecting (2.5) with $e \lambda^i \lambda^K$ we get

$$\lambda^K \nabla_K \mu_j = 2 t_j - e \sigma \mu_j \tag{2.6}$$

where $\sigma = \lambda^m t_m = \lambda^m \lambda^n \nabla_m \mu_n$.

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[2]) Proof of (2.3): Since $e a_{jl} = \mu_j a_{Kl} \lambda^K$, $e a_{jl} \lambda^l = \mu_j a_{Kl} \lambda^K \lambda^l$ or, $e a_{jl} \lambda^l = \mu_j \varrho \therefore a_{jl} \lambda^l = e \mu_j \varrho$. Hence $e a_{jl} = \mu_j e \mu_l \varrho = e \mu_j \mu_l$.

Again from (2.5)

$$\nabla_K \mu_i = e(\mu_i t_K + 2\mu_K t_i) - 2\sigma \mu_i \mu_K. \tag{2.7}$$

Put

$$p_i = t_i - e\sigma\mu_i.$$

Then (2.7) can be expressed as

$$\nabla_K \mu_i = e(\mu_i t_K + 2p_i \mu_K). \tag{2.8}$$

Again from $\nabla_l \nabla_K R_{ij} = \beta_l \nabla_K R_{ij} + a_{Kl} R_{ij}$, we get

$$\begin{aligned} &\beta_l(\mu_j \nabla_K \mu_i + \mu_i \nabla_K \mu_j) + \varrho \mu_i \mu_j \mu_K \mu_l = \\ &= \mu_j \nabla_l \nabla_K \mu_i + \mu_i \nabla_l \nabla_K \mu_j + \nabla_K \mu_i \nabla_l \mu_j + \nabla_l \mu_i \nabla_K \mu_j. \end{aligned} \tag{2.9}$$

Transvecting (2.9) with λ^i and λ^j we get

$$\beta_l(t_K \mu_j + e\nabla_K \mu_j) + e\varrho \mu_j \mu_K \mu_l = \mu_j \lambda^i \nabla_l \nabla_K \mu_i + e\nabla_l \nabla_K \mu_j + t_K \nabla_l \mu_j + t_l \nabla_K \mu_j \tag{2.10}$$

and

$$2e\beta_l t_K + \varrho \mu_K \mu_l - 2t_K t_l = 2e\lambda^j \nabla_l \nabla_K \mu_j. \tag{2.11}$$

Using (2.11) we get from (2.10)

$$e\nabla_l \nabla_K \mu_j = e\beta_l \nabla_K \mu_j + \tfrac{1}{2} e\varrho \mu_j \mu_K \mu_l + e\mu_j t_K t_l - t_K \nabla_l \mu_j - t_l \nabla_K \mu_j. \tag{2.12}$$

Using (2.12) and (2.8) we have from (2.9) by straightforward calculations

$$8 p_i p_j \mu_K \mu_l = 0. \tag{2.13}$$

Hence $p_i = 0$, i. e.,

$$t_i = e\sigma\mu_i.$$

Therefore $\nabla_K \mu_i = \sigma \mu_i \mu_K$. From $R_{ij} = e\mu_i \mu_j$ we get $\nabla_K R_{ij} = \varphi_K R_{ij}$, where $\varphi_K = 2\sigma\mu_K$. From this we get

$$\nabla_l R_{hijK} = \varphi_l R_{hijK}. \tag{2.14}$$

If $\varphi_l = 0$, the space becomes flat which is impossible by assumption. Hence $\varphi_l \neq 0$. This means that $\sigma \neq 0$. Thus incidentally we see that λ^i and μ^i introduced in this section must not satisfy the equation $\lambda^i \lambda^j \nabla_i \mu_j = 0$.

We now state the following theorem:

Theorem 4. *If there exists a conformally flat $G(^2K_n)$ with symmetric recurrence tensor and zero scalar curvature then the space is a recurrent space.*

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