On a generalisation of the Tammes problem

By J. MOLNÁR (Budapest)

To professor A. Rapcsák on his sixtieth birthday

The aim of this paper is to call attention to the relation which seem to exist between a generalisation of the Tammes problem and the structure of some viruses.

The biologist TAMMES [23], studying the distribution of the openings on the surface of spherical pollen grains, noted that these openings are dispersed as far as possible from each other. He raised the following problem:

To distribute n points on the sphere so that the least distance among the points should be as great as possible.

Of course this problem is the same as the determination of the largest radius of n equal circles which can be packed on a spherical surface without overlapping.[1])

In a note GOLDBERG [11] compared his results on axially symmetric circle-packing, for the cases $n=32$ and $n=42$, with the structure of some (small) polyhedral viruses having icosahedral symmetry. GOLDBERG's examples showed that in this cases the considered virus structures do not constitute the solution of the Tammes problem and he concluded:

"Since the symmetries observed in viruses are not the solution of the problem being considered, they must result from some other problem the mathematical formulation of which is unknown, at present."

In earlier notes [17], [18], [19] we have introduced and applied the notion of space-claim and we have characterized with extremal property (densest packing) among others in euclidean plane the homogeneous[2]) circle-packings of SINOGOWITZ [22], of NIGGLI [21] and the homogenous circle packing of DOMINYÁK on the sphere.[3])

In our investigations we have considered essentially space-claim circle-systems and the problem treated in its most simple form can be formulated as follows:

What is the densest packing of equal circles C_i which can be packed on a spherical surface without overlapping and having the additional property that each circle

[1]) The literature of the Tammes problem is very rich. See references: FEJES-TÓTH [4], [5], [6], COXETER [2], HEPPES and MOLNÁR [13], GOLDERG [10], [11], MOLNÁR [17], [18], [19].

[2]) In euclidean plane or on the sphere a circle-system is said to be homogeneous if for two arbitrary circles of the system there is an isometry which transforms these circles into one another but the whole system remains unchanged. The enumeration of homogeneous circle-packings on the sphere (unpublished) is due to I. DOMINYÁK.

[3]) On the sphere the density of a set of circles is defined as the quotient of the total area of the circles and the surface area of the sphere.

C_i has a tangent circle C_i^* of given radius — called space-claim — which has no common point with the interior of C_i but two different space-claims C_i^* and C_j^* may have common points.

Among the extremal circle-packings as solutions of this problem we obtain the regular ones and the homogeneous of type (3, 3, 3, 4) and (3, 3, 3, 5) (Fig. 1), more precisely, the centers of the circles determine the archimedean tessellations (3, 3, 3, 4) and (3, 3, 3, 5).[4])

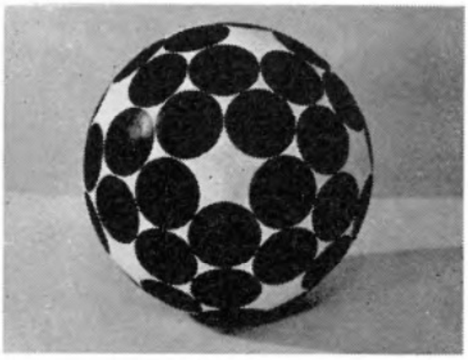

Fig. 1.

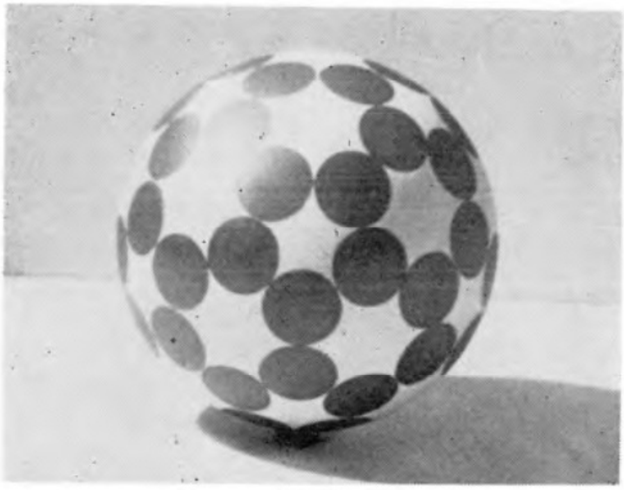

Fig. 2.

This problem, of course, is a generalisation of the Tammes problem, containing supplemantary condition. The space-claim C_i^*, in general, can be an arbitrary set of points, whose components, in some cases, can be moved.[5])

If the space-claim of the circle C_i is a domain composed of two congruent circles C_i^{*1} and C_i^{*2} tangent to C_i, as solution of the above problem we obtain the homogeneous circle-packings of types (3, 4, 3, 4) and (3, 5, 3, 5). It seems that the virus Bacteriophage φ X174 has a structure of type (3, 5, 3, 5) (Fig. 3, 4).[6])

Fig. 3.

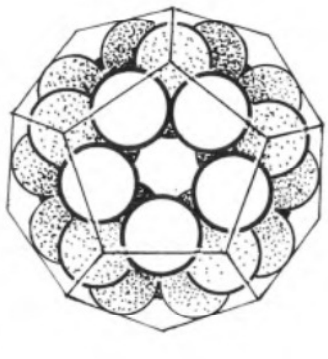

Fig. 4.

Concerning the structure of small "spherical" viruses we suppose that in the construction of the shell (capsid) the economical principle is valid[7]), more precisely, the shell is a densest packing of identical building units disposing of well determined space-claim.[8])

We now define in a set of circles $\{C_i\}$ the *Dirichlet* cell D_i of a circle C_i as the set of points which have no preater power with respect to the respective circle than with respect to any other circle. It is easy to define the *Dirichlet* cell also in the case if C_i is itself a set of circles. If $C_i = \cup c_k$ we take as *Dirichlet* cell $D_i = \cup d_k$, where d_k is the *Dirichlet* cell of c_k.[9])

We obtain a further generalisation of the Tammes problem if we consider instead of the circle C_i a set of circles.

To illustrate this case we give an example which reminds us of the structure of the Turnip Yellow Mosaic Virus (TYMV) (Fig. 5).[10]) In this virus the protein molecules of the capsid arranged in 32 capsomers (12 pentamers and 20 hexamers) form a structure with a corresponding circle-packing Π of two kinds of circles of radii $r_5 \approx 6°52'13''$ and $r_6 \approx 7°41'30''$ (Fig. 6). These 180 circles of Π can be grouped in 60 equivalent subunits (trimers), so that in each subunit there are three tangent circles, one of radius r_5 and two of radius r_6 (Fig. 7). If we take as building unit the

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[4]) A tessellation is a set of polygons joining along whole sides so as to fill the plane without interstices and without overlapping. A tessellation with regular faces and equivalent vertices is said to be uniform. The archimedean tessellations are uniform.

[5]) MOLNÁR [17], [19]. Fig. 2 illustrates an extremal homogeneous circle-packing (5, 6, 6) in which each circle has a space-claim of three circles of two kinds.

[6]) See the article of R. W. HORNE in HAYNES and HANAWALT [12], 124. p.

[7]) Compare our supposition with the assumption of CRICK and WATSON [3] that the shell of small "spherical" viruses are probably built from a number of identical protein subunits packed symmetrically.

[8]) It is possible that the space-claim of the building unit serves to accomplish some special functions.

[9]) MOLNÁR [20].

trimer and consider it as C_i and as the corresponding space-claim C_i^* the three tangent circles, one of radius $\varrho_5 \approx 6°52'22''$ inscribed in the pentamer and two of radius $\varrho_6 \approx 7°33'21''$ inscribed in hexamers (Fig. 7) we obtain as extremal packing the circle-packing Π corresponding to TYMV.

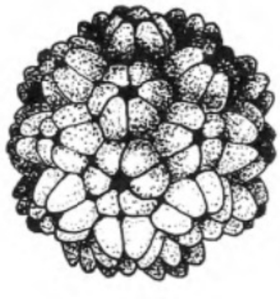

Fig. 5.

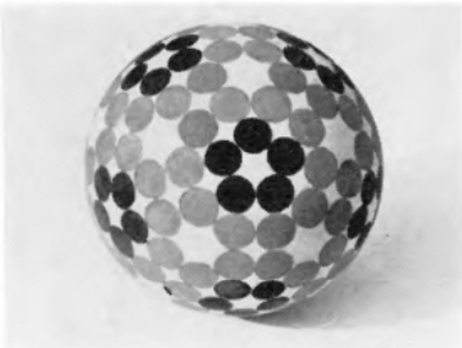

Fig. 6.

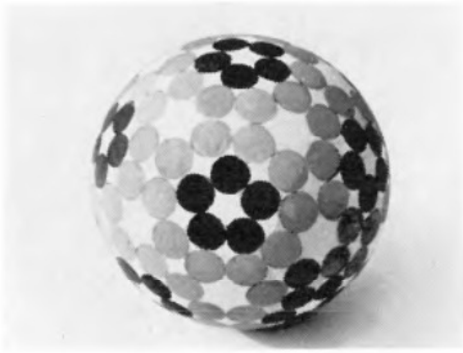

Fig. 8.

The idea of the proof, that the considered circle-packing Π is extremal, is the same as the one we have used already in our earlier papers. In fact, denoting by D_i the *Dirichlet* cell corresponding to C_i and by H_i the convex hull of the centers of the components of the space-claim C_i^* (in Fig. 7 the isosceles triangle), we obtain, of course, $\frac{C_i}{D_i} \leqq \frac{C_i}{H_i}$, therefore the density of the system $\{C_i\}$ with respect to the surface-area of the sphere is no greater than $\frac{C_i}{H_i}$. This density $\frac{C_i}{H_i}$ is realised by the circle-packing Π.

An analogous case is represented by the extremal circle-packing Π^* corresponding to the Bovine Enterovirus (Fig. 8).[11]) In this circle-packing the pentamers are rotated by an angle of $\frac{\pi}{5}$ with respect to the circle-packing Π corresponding to TYMV. The space-claim C_i^* of the trimer C_i has four components and the convex hull H of the centers of the components of C_i^* is a deltoid (Fig. 9).

It is interesting to remark that in the caces of circle-packings Π and Π^* it is only the economical principle (densest packing) that leads us to the symmetrical

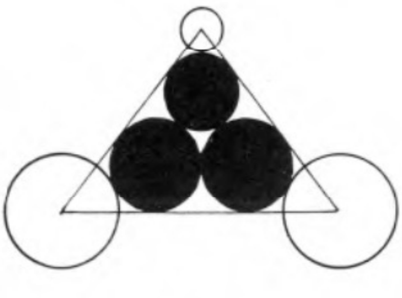

Fig. 7.

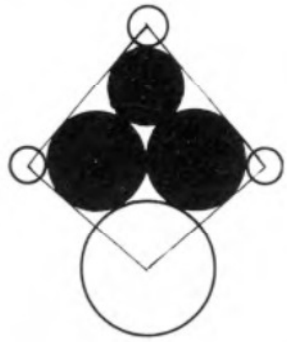

Fig. 9.

structure. In other words, the symmetry of the packing is a direct consequence of the economical principle.

At the end we make a remark concerning the solid circle-packings.[12])

A circle-packing is said to be solid if no finite subset of the circles can be rearranged so as to obtain a packing not congruent to the original one. *Fejes-Tóth* proved among others that the circle-packing composed by the incircles of the tessellation (5, 6, 6) is a solid circle-packing.

If we consider instead of the incircles of (5, 6, 6) the corresponding pentamers and hexamers of Π resp. of Π^*, these, of course, do not form a solid circle-packing with respect to the pentamers resp. hexamers as building units. But if we take as building units the trimers of Π resp. of Π^* with its space-claims, it is not difficult to show that Π resp. Π^* are solid packings. In this sense the circle-packings Π resp. Π^* corresponding to TYMV resp. to the Bovine Enterovirus are solid packings.

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