On non-linear covariant derivative in the General Geometry of Paths

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To Professor A. Rapcsák on his 60th birthday

§ 1. Introduction

A well-known generalization of Metrical Geometries is the so called General Geometry of Paths (DOUGLAS [1], KNEBELMAN [2], RAPCSÁK [3]), defined by introducing a system of differential equations of form

$$\frac{d^2x^i}{dt^2}+2G^i\left(x^j, \frac{dx^j}{dt}\right)=0, \tag{1}$$

where $G^i(x^j, \dot{x}^j)$ are given functions of the line-elements of an underlying manifold M_n. It is also assumed that these functions are positively homogeneous of the second degree in the $\dot{x}^j$'s and that the transformation properties of the G^i's are such as they leave the equations (1) invariant.

A basic method of the study of differential geometric spaces is the use of tensor calculus by introduction a covariant derivative. In the General Geometry of Paths this is possible in two ways: one is the BERWALD's connection theory [4] the other is the theory of the non-linear connection (FRIESECKE [5], BORTOLOTTI [6], VAGNER [7], KAWAGUCHI [8], BARTHEL [9], [10], KANDATU [11], TAMÁSSY [12]).

The covariant derivative of a non-linear connection (in the following: non-linear covariant derivative) is defined by

$$\nabla_k X^i = \frac{\partial X^i}{\partial x^k}+G^i_k(x^j, X^j), \tag{2}$$

where $G^i_k = \dfrac{\partial G^i}{\partial \dot{x}^k}$ (G^i-s are the functions in (1)).

In the following we shall give a characterization of the non-linear covariant derivative in the case, when homogeneity of the first degree in the $\dot{x}^j$'s of the functions G^i_k's in (2) is assumed (generally the positive homogeneity is assumed only).

In § 2. we shall give a global definition of a non-linear connection by a modification of the KOSZUL's definition of a linear connection. In [11] A. KANDATU has given a definition in a similar way, but we show in Theorem 1. that KANDATU's postulate (c) follows from the other ones.

In § 3. we shall define the connection map of the non-linear connection, and by making use of this map we shall show that the non-linear connection uniquely determines a horizontal distribution on TM. Our construction is similar to that of P. DOMBROWSKI in the case of a linear connection (cf. App. (IV) of [13] and § 2.4 of [14]). It is to be noted that a non-linear connection was defined as a horizontal distribution on the tangent bundle in the first global formulation of the theory of non-linear connections (BARTHEL [10]).

In § 4. we shall give a functional analytic characterization of a smooth non-linear covariant derivative. The smoothness of a non-linear connection is studied by GÄHLER [15] in an other approach.

Notation. The manifolds and maps of class C^∞ are called smooth. If M is a smooth manifold, then T_pM will denote the tangent space of M at the point $p \in M$, TM will denote the tangent bundle, $T'M \subset TM$ is the open subset of all nonzero tangent vectors. FM will denote the ring of the smooth functions on M. The FM-module of the smooth vectorfields on M is denoted by $S^\infty(TM)$, the FM-module of all vectorfields on M will be denoted by $S(TM)$. If $f\colon M \to N$ is a smooth map of the smooth manifold M into the smooth manifold N, then df will denote the differential of f, $df: TM \to TN$.

§ 2. The non-linear covariant derivative

Definition. We define the covariant derivative of a non-linear connection on the smooth manifold M as a mapping

$$\nabla: (X, Y) \in S^\infty(TM) \times S^\infty(TM) \to \nabla_X Y \in S(TM)$$

satisfying the following conditions:

$$(1)\qquad \begin{cases} 1^\circ \ \nabla_X fY = (Xf)Y + f\nabla_X Y; \\ 2^\circ \ \nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y; \\ 3^\circ \ \nabla_{fX} Y = f\nabla_X Y; \\ 4^\circ \ \text{if } p \in M \text{ and } Y_1|_p = 0 \text{ then } \nabla_X (Y_1+Y_2)|_p = \nabla_X Y_1|_p + \nabla_X Y_2|_p, \end{cases}$$

where X, X_1, X_2, Y, Y_1 and Y_2 are arbitrary elements of $S^\infty(TM)$ and f is an arbitrary element of FM.

Let U be a coordinate neighbourhood of the manifold M with the local coordinates $x^1, \ldots, x^n$ and let the local representation of the vector fields X and Y be $X^i \dfrac{\partial}{\partial x^i}$ and $Y^i \dfrac{\partial}{\partial x^i}$, respectively.

Theorem 1. *The non-linear covariant derivative $\nabla_X Y$ has the following local representation:*

$$(2)\qquad \nabla_X Y = X^i \left[\frac{\partial Y^j}{\partial x^i} + G_i^j(x^k, Y^k) \right] \frac{\partial}{\partial x^j},$$

where the functions $G_i^j(x^1, \dots, x^n; y^1, \dots, y^n)$ *are defined on* $x(U)\times R^n$ *and are homogeneous of degree* 1 *with respect to* $(y^1, \dots, y^n)$.

PROOF. We identify the restriction $U\subset M$ of the tangent bundle $\{TM, \pi, M\}$ with the trivial bundle $\{x(U)\times R^n, \pi, x(U)\}$ by identifying the vector fields $\frac{\partial}{\partial x^i}$ on U with n linearly independent vectors of the vector space R^n. Then we can consider the vector fields X and Y on U as maps $x(U)\to R^n$.

We put:

$$G_i^*(Y) = \nabla_{\frac{\partial}{\partial x^i}} Y - \frac{\partial Y}{\partial x^i}. \tag{3}$$

$G_i^*(Y)$ is a vector field on U, and so it can be written as a linear combination of the $\frac{\partial}{\partial x^i}$-s:

$$G_i^*(Y) = G_i^{*j}(Y)\frac{\partial}{\partial x^j}. \tag{4}$$

Hence we have by (3) and (4)

$$\nabla_{\frac{\partial}{\partial x^i}} Y = \left[\frac{\partial Y^j}{\partial x^i} + G_i^{*j}(Y)\right]\frac{\partial}{\partial x^j}.$$

The postulates 2° and 3° of (1) imply that

$$\nabla_X Y = X^i\left[\frac{\partial Y^j}{\partial x^i} + G_i^{*j}(Y)\right]\frac{\partial}{\partial x^j}.$$

Now, we show that the vector-valued functions $G_i^{*j}(Y)$ on $x(U)\subset R^n$ can be represented in the form

$$G_i^{*j}(Y)(x^1, \dots, x^n) = G_i^j\big(x^1, \dots, x^n;\ Y^1(x^1, \dots, x^n), \dots, Y^n(x^1, \dots, x^n)\big),$$

where the functions $G_i^j(x^1, \dots, x^n; y^1, \dots, y^n)$ are uniquely determined by the non-linear covariant derivative ∇.

Let be $p\in U$, and stipulate for the vector fields Y_1 and Y_2 that $Y_1|_p = Y_2|_p$. Than we show that $G_i^{*j}(Y_1)|_p = G_i^{*j}(Y_2)|_p$. We apply postulate 4° of (1) for the vector fields Y_1 and $Y_2 - Y_1$ at the point $p\in U$:

$$\nabla_{\frac{\partial}{\partial x^i}}\big(Y_1 + (Y_2 - Y_1)\big)\big|_p = \nabla_{\frac{\partial}{\partial x^i}} Y_1|_p + \nabla_{\frac{\partial}{\partial x^i}}(Y_2 - Y_1)\big|_p.$$

We again apply postulate 4°:

$$\nabla_{\frac{\partial}{\partial x^i}} \sum_{j=1}^{n}\left[(Y_2^j - Y_1^j)\frac{\partial}{\partial x^j}\right]\Bigg|_p = \sum_{j=1}^{n}\nabla_{\frac{\partial}{\partial x^i}}(Y_2^j - Y_1^j)\frac{\partial}{\partial x^j}\bigg|_p$$

(here we do not sum for the index j, as Einstein's convention would require). By postulate 1° we can write:

$$\nabla_{\frac{\partial}{\partial x^i}}\left[(Y_2^j - Y_i^j)\frac{\partial}{\partial x^j}\right]\Bigg|_p = \frac{\partial(Y_2^j - Y_1^j)}{\partial x^i}\frac{\partial}{\partial x^j}\bigg|_p + (Y_2^j - Y_1^j)\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\bigg|_p.$$

Since we have $Y_1|_p = Y_2|_p$, the last term of this equation vanishes and so we have:

$$\nabla_{\frac{\partial}{\partial x^i}} (Y_1 + (Y_2 - Y_1))|_p = \nabla_{\frac{\partial}{\partial x^i}} Y_1|_p + \frac{\partial (Y_2^j - Y_1^j)}{\partial x^i} \frac{\partial}{\partial x^j}\bigg|_p .$$

According to (3), this is equivalent to the desired equality:

$$G_i^{*j}(Y_2) \frac{\partial}{\partial x^j}\bigg|_p = G_i^{*j}(Y_1) \frac{\partial}{\partial x^j}\bigg|_p .$$

Now, we can define the functions $G_i^j(x^1, \ldots, x^n; y^1, \ldots, y^n)$. Let $(x_0^1, \ldots, x_0^n; y_0^1, \ldots, y_0^n)$ be an arbitrary, fixed element of $x(U) \times R^n$.

Assume that the point $p \in U$, coincides with that having coordinates $x_0^1, \ldots, x_0^n$. Let $Y^i : x(U) \to R^1$ be arbitrary functions such that $Y^i(x_0^1, \ldots, x_0^n) = y_0^i$. Let Y be the vectorfield on U having coordinate functions $Y^i(x^1, \ldots, x^n)$. Then we define the functions:

$$G_i^j(x_0^1, \ldots, x_0^n; y_0^1, \ldots, y_0^n) = G_i^{*j}(Y)|_p .$$

According to the results obtained above this is a correct definition.

We have to that the functions G_i^j are homogeneous with respect to the variables y^k-s.

We apply property 1° of (1):

$$G_i^*(fY) = \nabla_{\frac{\partial}{\partial x^i}} (fY) \frac{\partial (fY)}{\partial x^i} = \left(\frac{\partial f}{\partial x^i} Y + f \nabla_{\frac{\partial}{\partial x^i}} Y\right) - \left(\frac{\partial f}{\partial x^i} Y + f \frac{\partial Y}{\partial x^i}\right) = f G_i^*(Y).$$

From this the desired property follows.

§ 3. The connection map of the nonlinear connection

Let us consider the second tangent bundle of M: $\{TTM, \pi_1, TM\}$ ($\pi_1 : TTM \to TM$ is the projection map). $\{VTM, \pi_1, TM\}$ denotes the vertical subbundle of the second tangent bundle, i.e. the fiber $V_w TM$ of the vertical subbundle on the point $w \in TM$ is the set of the tangent vectors to the tangent space $T_{\pi(w)} M$. In other words, it is the kernel of the map $d\pi_w : T_w TM \to T_{\pi(w)} M$.

It is well known that the non-vertical vectors of TTM are of form $dY(v)$, where $Y \in S^\infty(TM)$, $v \in TM$ (cf. § 2.4 of [14]).

Let be $q \in M$, $w \in T_q M$. Let $\iota : T_q M \to TM$ denotes the inclusion map, and $I_w : T_q M \to T_w(T_q M)$ denotes the canonical vector space isomorphism of the vector space $T_q M$ onto its tangent space at $w \in T_q M$.

Definition. The connection map $K : TTM \to TM$ of the non-linear connection ∇ is defined by

$$K(a) = \begin{cases} \nabla_v Y & \text{if a has form } a = dYv, \\ I_w^{-1} d\iota^{-1}(a) & \text{if } a \in V_w TM. \end{cases}$$

Let $x^1, \ldots, x^n$ be the local coordinates in a neighbourhood $U \subset M$, and denote the induced coordinates on the neighbourhood $TU \subset TM$ by $\bar{x}^1, \ldots, \bar{x}^{2n}$. It easily follows from Theorem 1 that the connection map K has the following local representation:

$$K(a) = \sum_{k=1}^{n} \left(a^{n+k} + a^i G_i^k(\bar{x}^1(w), \dots, \bar{x}^{2n}(w))\right) \frac{\partial}{\partial x^k}\bigg|_q, \tag{5}$$

where $a \in T_w TM$, $q = \pi(w)$, $a = \sum_{r=1}^{2n} a^r \frac{\partial}{\partial \bar{x}^r}$, $w = w^j \frac{\partial}{\partial x^j}$. (Cf. formula (5) in § 2.4 of [14].)

It follows from the local representation of the connection map that the definition of this map is independent of the choice of the representation $a = dY(v)$ of the vector $a \in TTM$ and that the connection map is a linear map in the second tangent space.

Theorem 2. *The non-linear connection ∇ on the manifold M uniquely determines the horizontal subbundle $\{HTM, \pi_1, TM\}$ of the second tangent bundle $\{TTM, \pi_1, TM\}$, which has the following properties:*

1° $H_v TM \oplus V_v TM = T_v TM$ holds for any $v \in TM$,

2° $dcH_v TM = H_{cv} TM$ holds for any real number c,

where the real number c is considered as the homothety $v \in TM \to c \cdot v \in TM$.

Proof. The horizontal subbundle $\{HTM, \pi_1, TM\}$ is defined as the kernel of the connection map K i. e. for any $v \in TM$

$$H_v TM = \{a \in T_v TM : K(a) = 0\}.$$

Property 1° of the horizontal subbundle follows from the local representation (5) of the connection map K immediately (cf. § 2.4 of [14]). Property 2° follows similarly from (5) and from the fact that the functions $G_k^i(x^1, \dots, x^n; y^1, \dots, y^n)$ are homogeneous with respect to $(y^1, \dots, y^n)$ (Theorem 1).

§ 4. The smoothness of the non-linear connection

Definition. We say that a non-linear connection ∇ is *smooth* if the functions $G_k^i(x; y)$ occurring in its local representation are of class C^∞ on the set $(x(U) \times R^n) \setminus (x(U) \times \{0\})$ and continuous on their whole domain $(x(U) \times R^n)$.

It should be noted that the assumption of the continuity on $(x(U) \times \{0\})$ is not an essential condition, since it is easy to show that continuity on the set $(x(U) \times R^n) \setminus (x(U) \times \{0\})$ and homogeneity with respect to the variables $(y^1, \dots, y^n)$ of the functions in question imply the existence of a continuous extension of them to $(x(U) \times R^n)$ having the value 0 on $(x(U) \times \{0\})$.

We summarize some concepts and notations related to the Gâteaux's differentiability of maps of topological vector spaces (cf. [16]).

Let H and K be topological vector spaces and $T: H \to K$ a map. T is differentiable in Gâteaux's sense at the point $Y \in H$ if the limit:

$$\delta_Z T(Y) = \lim_{t \to 0} \frac{T(Y + tZ) - T(Y)}{t}$$

exists for every $Z \in H$. We then say that $\delta_Z T(Y) \in H$ is the (Gâteaux) derivative in the direction $Z \in H$ at the point $Y \in H$, of the map T. Higher derivatives are defined by recurrence: T is m-times differentiable at the point $Y \in H$ if it is $(m-1)$ time differentiable and the limit:

$$\delta_{(Z_1, \dots, Z_m)} T(Y) = \lim_{t \to 0} \frac{\delta_{(Z_1, \dots, Z_{m-1})} T(Y + tZ_m) - \delta_{(Z_1, \dots, Z_{m-1})} T(Y)}{t}$$

exists for every $Z_1, \dots, Z_m \in H$. We then say that $\delta_{(Z_1, \dots, Z_m)} T(Y) \in H$ is the mth derivative in the direction $(Z_1, \dots, Z_m)$ at the point $Y \in H$ of the map T. It is well-known that the set of the smooth vectorfields $S^\infty(TM)$ (and similarly also $S(TM)$) on a manifold M forms a topological vector space with respect to pointwise convergence.

Let us denote the set of the nonvanishing global smooth vectorfields on M by $S'^\infty(TM)$, and the set of all nonvanishing cross sections of TM by $S'(TM)$.

Lemma. *Let ∇ be a smooth non-linear connection on the manifold M. Then for any nonvanishing vector field $Y \in S'^\infty(TM)$, and for any $X \in S^\infty(TM)$ the map $\nabla_X : S^\infty(TM) \to S(TM)$ is infinitely many times differentiable at the point Y.*

PROOF. Let $U \subset M$ be a coordinate neighbourhood with local coordinates $x^1, \dots, x^n$. We compute the limit

$$\lim_{t \to 0} \frac{\nabla_X (Y + tZ_1) - \nabla_X Y}{t}$$

in local representation by using Theorem 1:

$$\lim_{t \to 0} X^k \frac{\left[\dfrac{\partial (Y^i + tZ_1^i)}{\partial x^k} + G_k^i(x^j;\, Y^j + tZ_1^j)\right] - \left[\dfrac{\partial Y^i}{\partial x^k} + G_k^i(x^j;\, Y^j)\right]}{t} \frac{\partial}{\partial x^i} =$$

$$= X^k \left[\frac{\partial Z_1^i}{\partial x^k} \frac{\partial}{\partial x^i} + \lim_{t \to 0} \frac{G_k^i(x^j;\, Y^j + tZ_1^j) - G_k^i(x^j;\, Y^j)}{t} \frac{\partial}{\partial x^i}\right] =$$

$$= \left[\frac{\partial Z_1^i}{\partial x^k} + \frac{\partial G_k^i}{\partial y^r} Z_1^r\right] X^k \frac{\partial}{\partial x^i}$$

since the functions G_k^i are on $U \times (R^n \setminus \{0\})$ of the class C^∞ and since $Z_1 = Z_1^r \dfrac{\partial}{\partial x^r}$ is a smooth vectorfield on U.

We determine the second derivative of the map ∇_X in the direction (Z_1, Z_2):

$$\lim_{t \to 0} \frac{\left[\dfrac{\partial Z_1^i}{\partial x^k} + \dfrac{\partial G_k^i(x^j;\, Y^j + tZ_2^j)}{\partial y^r} Z_1^r\right] - \left[\dfrac{\partial Z_1^i}{\partial x^k} + \dfrac{\partial G_k^i(x^j;\, Y^j)}{\partial y^r} Z_1^r\right]}{t} X^k \frac{\partial}{\partial x^i} =$$

$$= \frac{\partial^2 G_k^i(x^j;\, Y^j)}{\partial y^r \, \partial y^s} Z_1^r Z_2^s X^k \frac{\partial}{\partial x^i}.$$

Further analogous computation shows that the local representation of the mth derivative of the map ∇_X in the direction $(Z_1, \ldots, Z_m)$ is the following:

$$\frac{\partial^m G^i_k(x^j; Y^j)}{\partial y^{r_1} \ldots \partial y^{r_m}} Z_1^{r_1} Z_2^{r_2} \ldots Z_m^{r_m} X^k \frac{\partial}{\partial x^i}. \tag{6}$$

The formulas obtained prove the lemma.

Theorem 3. *Let be ∇ a non-linear connection on the smooth manifold M. ∇ is smooth if and only if for any open submanifold $U \subset M$ and for any smoth vectorfield X the restriction of the map ∇_X to $S^\infty(TU)$ has the following property: For any non-vanishing vectorfield $Y \in S'^\infty(TU)$ the map $\nabla_X: S^\infty(TU) \to S(TU)$ is infinitely many times differentiable at Y and the derivatives belond to $S^\infty(TU)$.*

Proof. The necessity of the given condition follows from the preceding lemma. Let be U a coordinate-neighbourhood, and let $(x_0^1, \ldots, x_0^n;\ y_0^1, \ldots, y_0^n) \in U \times \times (R^n \setminus \{0\})$. Let us consider the locally constant vectorfield $Y = y_0^i \dfrac{\partial}{\partial x^i}$ on U. Then $\nabla_X Y$ is a smooth vectorfield on U. In wiev of the local representation of $\nabla_X Y$, it follows from this that the functions G^i_k are infinitely many times differentiable with respect to the variables $(x^1, \ldots, x^n)$ at the point $(x_0^1, \ldots, x_0^n; y_0^1, \ldots, y_0^n)$.

Let us consider the vectorfields $Z_j = \dfrac{\partial}{\partial x^j}$ on U. Then by (6), the functions

$$\frac{\partial^m G^i_k(x_0^1, \ldots, x_0^n; y_0^1, \ldots, y_0^n)}{y^{s_1} y^{s_2} \ldots y^{s_m}} X^k \tag{7}$$

exist for any $1 \leqq s_1, \ldots, s_m \leqq n$. This proves that the functions G^i_k are infinitely many times differentiable with respect to the variables $(y^1, \ldots, y^n)$ at the point $(x_0^1, \ldots, x_0^n; y_0^1, \ldots, y_0^n)$.

Moreover, the functions (7) are of class C^∞ by our assumptions. It follows that the mixed derivatives of the functions G^i_k with respect to the variables x^i and y^i also exist at the point $(x_0^1, \ldots, x_0^n; y_0^1, \ldots, y_0^n)$.

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