Investigation in the power sum theory IV

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To the sixtieth birthday of A. Rapcsák

1. In this and next two papers of this series we will discuss an oscillation theorem referring to ordinary differential equations. Actually the method in title will be concerned in papers IV. and V.; in the sixth paper the general oscillation theorem will be proved based on the results of IV. and V. and has no direct connection with the method. Paper VI. could not be included either into paper IV. or V. without making it too long.

Let the real constants $a_0, a_1, \ldots, a_{n-1}$ be such that the zeros of the equation

$$z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0 \tag{1.1}$$

are outside the strip

$$|\mathrm{Im}\, z| \leq \Lambda \tag{1.2}$$

with a $\Lambda > 0$. Then we assert the

Theorem 1. *All $y(t) \not\equiv 0$ real solutions of the linear differential equations*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0 \tag{1.3}$$

change sign in every real interval of length

$$> \frac{n\pi}{2\Lambda}. \tag{1.4}$$

We assert that Theorem 1 is bestpossible for all n's (which are necessarily even $n=2k$). Let us consider namely the equation

$$\sum_{\nu=0}^{k} \binom{k}{\nu} y^{(2\nu)}(t) = 0. \tag{1.5}$$

Here the charecteristic equation is $(1+\lambda^2)^k$ i.e. $\Lambda = 1$ and the roots are $\lambda = \pm i$ with multiplicity k. Hence the functions $t^\nu e^{\pm it}$ $(\nu = 0, 1, \ldots, k-1)$ and also $t^\nu \sin t$ $(\nu = 0, 1, \ldots, k-1)$ are solutions of (1.5) and thus also the function

$$y^*(t) = \left(1 - \frac{t}{\pi}\right)\left(1 - \frac{t}{2\pi}\right) \ldots \left(1 - \frac{t}{(k-1)\pi}\right) \sin t.$$

But this function does not change sign for

$$0 \leqq t \leqq k\pi = \frac{n\pi}{2} = \frac{n\pi}{2\Lambda}$$

indeed.

2. The proof of Theorem 1 is based on a modified form of the so called onesided theorem referring to generalised powersums

$$g(\nu) = \sum_{j=1}^{n} b_j z_j^\nu \quad \nu \geqq 0, \text{ integer} \tag{2.1}$$

where b_j's are arbitrary fixed complex numbers and z_j's fixed complex numbers satisfying

$$\min_j |z_j| = 1 \tag{2.2}$$

and also satisfying the argumentum restrictions

$$\varkappa \leqq |\text{arc } z_j| \leqq \pi \quad (j = 1, \dots, n) \tag{2.3}$$

with a $0 < \varkappa \leqq \frac{\pi}{2}$ say. Let $m \geqq 0$ and integer. Then the onesided theorem in question*) asserts the existence of integer ν_1 and ν_2 so that

$$m+1 \leqq \nu_1, \quad \nu_2 \leqq m + n\left(3 + \frac{\pi}{\varkappa}\right) \tag{2.4}$$

and the inequalities

$$\text{Re } g(\nu_1) \geqq \frac{1}{5n} |\text{Re } g(0)| \left(\frac{n}{27(m+n)}\right)^{2n} \tag{2.5}$$

and

$$\text{Re } g(\nu_2) \leqq -\frac{1}{5n} |\text{Re } g(0)| \left(\frac{n}{27(m+n)}\right)^{2n} \tag{2.6}$$

hold. The modified form refers to such $g(\nu)$'s where beside the normalisation (2.2) and argument restriction (2.3) the "symmetry-condition" holds which means that the z_j-system is symmetric to the real axis i.e.

(2.7) *with z_j also $\bar{z}_j$ occures among the z_j's*

and

$$\text{coeffs. } (\bar{z}_j)^\nu = \overline{\text{coeffs. } z_j^\nu}. \tag{2.8}$$

For such $g(\nu)$ — which are real for integer ν's — we have

Theorem 2. *For all integer $m \geqq 0$ there are integer ν_1 and ν_2 so that*

$$m+1 \leqq \nu_1, \quad \nu_2 \leqq m + n\left(\frac{3}{2} + \frac{\pi}{2\varkappa}\right)$$

*) P. Turán, On an improvement of some new onesided theorems. *Acta Math. Hung.* **11.** (1960), 299—316.

and the inequalities

$$g(\nu_1) \geqq \frac{1}{3n}|g(0)|\left(\frac{n}{8\sqrt{e}(m+n)}\right)^n$$

and

$$g(\nu_2) \leqq -\frac{1}{3n}|g(0)|\left(\frac{n}{8\sqrt{e}(m+n)}\right)^n$$

hold.

First we shall deduce Theorem 1 from Theorem 2 and then we shall sketch the proof of Theorem 2.

3. Let $\alpha_1, \ldots, \alpha_n$ be different complex numbers satisfying the inequalities

$$\min_j \operatorname{Re} \alpha_j = 0 \tag{3.1}$$

$$\min_j |\operatorname{Im} \alpha_j| = l > 0. \tag{3.2}$$

Further the symmetry conditions S should be satisfied that with α_j also $\bar{\alpha}_j$ occurs among the α_j's and the numbers c_j be such that

$$\bar{\alpha}_j = \alpha_{j_1} \tag{3.3}$$

implies

$$c_{j_1} = \bar{c}_j. \tag{3.4}$$

Let further $d>0$ be so small that

$$\frac{dl}{n} \leqq \frac{\pi}{2} \tag{3.5}$$

and we choose in Theorem 2

$$m = 0, \quad z_j = e^{\frac{d}{n}\alpha_j} \quad (j = 1, \ldots, n) \tag{3.6}$$

and

$$g(\nu) = g_1(\nu) = \sum_{j=1}^{n} c_j e^{\frac{d}{n}\alpha_j \nu}. \tag{3.7}$$

(3.3)—(3.4) implies that the symmetry-restrictions (2.7)—(2.8) are satisfied; owing to (3.1) the same holds for (2.2). As to (2.3) we have

$$\operatorname{arc} z_j = \frac{d}{n} \operatorname{Im} \alpha_j$$

i. e.

$$\varkappa = \frac{dl}{n} \tag{3.8}$$

can be chosen. Then Theorem 2 gives the existence of integer ν_1 and ν_2 with

$$0 < \nu_1, \quad \nu_2 \leqq n\left(3 + \frac{\pi n}{2dl}\right)$$

i. e.

(3.9)
$$0 < \frac{dv_1}{n}, \quad \frac{dv_2}{n} \leqq 3d + \frac{\pi n}{2l}$$

so that

$$g_1(v_1) \geqq \frac{1}{3n} |\mathrm{Re}\, g_1(0)| 27^{-n}$$

$$g_1(v_2) \leqq -\frac{1}{3n} |\mathrm{Re}\, g_1(0)| 27^{-n}.$$

But owing to (3.9) this means that putting

(3.10)
$$G(t) = \sum_{j=1}^{n} c_j e^{\alpha_j t}$$

we have

$$\max_{0 < x \leqq 3d + \frac{\pi n}{2l}} G(t) \geqq \frac{1}{3n} |\mathrm{Re}\, G(0)| 27^{-n}$$

$$\min_{0 < x \leqq 3d + \frac{\pi n}{2l}} G(t) \leqq -\frac{1}{3n} |\mathrm{Re}\, G(0)| 27^{-n}$$

or with $d \to 0$

$$\max_{0 \leqq x \leqq \frac{\pi n}{2l}} G(t) \geqq \frac{1}{3n} |\mathrm{Re}\, G(0)| 27^{-n}$$

$$\min_{0 \leqq x \leqq \frac{\pi n}{2l}} G(t) \leqq -\frac{1}{3n} |\mathrm{Re}\, G(0)| 27^{-n}.$$

If a is an arbitrary real number, replacing c_j in (3.10) by $c_j e^{-\alpha_j a}$ the symmetry restriction (3.3)—(3.4) is not violated and we got the inequalities

$$\max_{a \leqq x \leqq a + \frac{\pi n}{2l}} G(t) \geqq \frac{1}{3n} |\mathrm{Re}\, G(a)| 27^{-n}$$

(3.11)

$$\min_{a \leqq x \leqq a + \frac{\pi n}{2l}} G(t) \leqq -\frac{1}{3n} |\mathrm{Re}\, G(a)| 27^{-n}.$$

Let now $y = y(t) \not\equiv 0$ be an arbitrary solution of the equation (1.3) with property (1.1)—(1.2). Then the a_j coefficients in (1.1) can be changed so to a_j^*'s that they remain real, the zeros α_j of

$$z^n + a_{n-1}^* z^{n-1} + \ldots + a_0^* = 0$$

are for any prescribed $\varepsilon > 0$ outside the strip

$$|\mathrm{Im}\, z| \leqq \Lambda - \varepsilon$$

and all simple. But then the solutions of the modified equation

$$v^{(n)}+a'_{n-1}v^{(n-1)}+\ldots+a'_0 v=0, \quad v(a)=y(a)$$

are of the form (3.10) and so (3.11) is with $l=\Lambda-\varepsilon$ applicable. Then trivial passage to limit gives for arbitrary real a the inequalities

$$\max_{a\leqq x\leqq a+\frac{\pi n}{2\Lambda}} y(t) \geqq \frac{1}{3n}|y(a)|27^{-n}$$

$$\min_{a\leqq x\leqq a+\frac{\pi n}{2\Lambda}} y(t) \leqq -\frac{1}{3n}|y(a)|27^{-n}.$$

If $y(a)\neq 0$ the proof of Theorem 1 is finished; if not then arbitraly close to a there are a' places with $y(a')\neq 0$ and again the theorem follows.

4. So we have to prove Theorem 2. We shall need (see 1. c.p. 305—308) the

Lemma 1. *Let $F(z)=1+a_1z+\ldots+a_Nz^N$ be an arbitrary polynomial with real coefficients and with all zeros outside the angle*

$$|\operatorname{arc} z|<\varkappa \quad \left(0<\varkappa\leqq\frac{\pi}{2}\right).$$

Then there is a polynomial $\varphi(z)$ (with real coefficients) so that

$$F(z)\varphi(z)=1+e_1z+e_2z+\ldots \tag{4.1}$$

is a polynomial of degree

$$\leqq \frac{N}{2}\left(1+\left[\frac{\pi}{\varkappa}\right]\right) \tag{4.2}$$

and with nonnegative coefficients. Supposing in addition that all zeros of $F(z)$ are outside the disc $|z|<1$ we have also the estimation

$$\sum_{\nu} e_\nu \leqq 2^N. \tag{4.3}$$

We shall also state the following theorem of Norlund as

Lemma 2. *Let $w_1, \ldots, w_\nu$ pairwise different komplex numbers outside of a closed rectifiable curve L and $g(z)$ be analytical outside and on L with $g(\infty)=0$. Then defining the (unique) polynomial $L_\nu(g)$ of degree $\leqq \nu-1$ by*

$$L_\nu(g)_{z=w_j}=g(w_j) \quad j=1,2,\ldots,\nu \tag{4.4}$$

and writing it in the Newtonian form

$$L_\nu(g)=d_0+d_1(z-w_1)+d_2(z-w_1)(z-w_2)+\ldots+d_{\nu-1}(z-w_1)\ldots(z-w_{\nu-1}) \tag{4.5}$$

the coefficients d_j are given by

$$d_j=\frac{1}{2\pi i}\int_L \frac{g(w)\,dw}{(w-w_1)(w-w_2)\ldots(w-w_{j+1})}. \tag{4.6}$$

5. Let m be an arbitrary integer $\geqq 0$, the z_j's satisfying (2.7), (2.2) and consider the polynomial $\pi_{n-1}(z)$ defined by

$$\pi_{n-1}(z_j) = z_j^{-m-1}, \quad j = 1, 2, \ldots, n. \tag{5.1}$$

Writing

$$\pi_{n-1}(z) = c_0^{(1)} + c_1^{(1)} z + \ldots + c_{n-1}^{(1)} z^{n-1} \tag{5.2}$$

the $c_\nu^{(1)}$ coefficients are real. We shall write $\pi_{n-1}(z)$ first in the form

$$\pi_{n-1}(z) = d_0 + d_1(z-z_1) + d_2(z-z_1)(z-z_2) + \ldots + d_{n-1}(z-z_1) \ldots (z-z_{n-1}); \tag{5.3}$$

the connection of $c_\nu^{(1)}$ and d_j coefficients is obviously

$$c_{n-1}^{(1)} = d_{n-1}$$

and for $\nu \leqq n-2$

$$c_\nu^{(1)} = d_\nu - d_{\nu+1} \sum_{1 \leqq j_1 \leqq \nu+1} z_{j_1} + d_{\nu+2} \sum_{1 \leqq j_1 < j_2 \leqq \nu+2} z_{j_1} \cdot z_{j_2} - \ldots .$$

Hence Lemma 2 gives choosing as L the circe $L_1\colon |w| = 1 - \dfrac{1}{2} \dfrac{n}{m+n} = \varrho$ the representation

$$c_\nu^{(1)} = \frac{1}{2\pi i} \int_{L_1} \frac{1}{w^{m+1}(w-z_1) \ldots (w-z_{\nu+1})} \left\{ 1 - \frac{1}{w - z_{\nu+2}} \sum_{1 \leqq j_1 \leqq \nu+1} z_{j_1} + \right. \\ \left. + \frac{1}{(w-z_{\nu+2})(w-z_{\nu+3})} \sum_{1 \leqq j_1 < j_2 \leqq \nu+2} z_{j_1} z_{j_2} - \ldots \right\} dw. \tag{5.4}$$

We may suppose without less of generality

$$|z_1| \leqq |z_2| \leqq \ldots \leqq |z_n|. \tag{5.5}$$

Then the absolute value of the expression in the curly bracket cannot exceed

$$1 + \binom{\nu+1}{1} \frac{|z_{\nu+2}|}{|z_{\nu+2}| - \varrho} + \binom{\nu+2}{2} \frac{|z_{\nu+2}|}{|z_{\nu+2}| - \varrho} \cdot \frac{|z_{\nu+3}|}{|z_{\nu+3}| - \varrho} + \ldots$$

$$\ldots + \binom{n-1}{n-\nu-1} \frac{|z_{\nu+2}|}{|z_{\nu+2}| - \varrho} \frac{|z_{\nu+3}|}{|z_{\nu+3}| - \varrho} \cdots \frac{|z_n|}{|z_n| - \varrho}$$

further

$$\frac{|z_j|}{|z_j| - \varrho} = 1 + \frac{\varrho}{|z_j| - \varrho} \leqq 1 + \frac{\varrho}{1-\varrho} = 2\frac{m+n}{n}$$

we get from (5.4)

$$|c_\nu^{(1)}| \leqq \frac{1}{\varrho^{m+1}}\left(2\frac{m+n}{n}\right)^{\nu+1}\left\{1+\binom{\nu+1}{1}2\frac{m+n}{n}+\right.$$

$$\left.+\binom{\nu+2}{2}\left(2\frac{m+n}{n}\right)^2+\ldots+\binom{n-1}{n-\nu-1}\left(2\frac{m+n}{n}\right)^{n-\nu-1}\right\} \leqq$$

$$\leqq \frac{1}{\varrho^{m+1}}\left(2\frac{m+n}{n}\right)^n\binom{n}{\nu+1}.$$

Using the value of ϱ we obtain the

Lemma 3. *For the (real) $c_\nu^{(1)}$ coefficients the inequality*

$$\sum_{\nu=0}^{n-1}|c_\nu^{(1)}| \leqq \left(4\sqrt{e}\,\frac{m+n}{n}\right)^n \tag{5.6}$$

holds.

6. We can now turns to the proof of Theorem 2. We may suppose without loss of generality $z_\mu \neq z_\nu$ for $\mu \neq \nu$. We construct a sequence of auxiliary polynomials. The first is the one in (5.1); we shall denote it by $f_1(z)$ let the second be

$$f_2(z) = \prod_{j=1}^{n}\left(1-\frac{z}{z_j}\right) = \sum_{\nu=0}^{n} c_\nu^{(2)} z^\nu \tag{6.1}$$

which has owing to (2.7) only real coefficients. Owing to (2.3) we can apply Lemma 1 with

$$F(z) = f_2(z), \quad N = n.$$

Then we get, the polynomial

$$f_3(z) = f_2(z)\varphi(z) = 1 + c_1^{(3)}z + \ldots \tag{6.2}$$

with nonnegative coefficients of degree

$$\leqq \frac{n}{2}\left(1+\left[\frac{\pi}{\varkappa}\right]\right) \tag{6.3}$$

so that

$$\sum_\nu c_\nu^{(3)} \leqq 2^n. \tag{6.4}$$

Next let us consider the fourth one

$$f_4(z) = f_3(z)(1+z+z^2+\ldots+z^{n-1})2\left(4\sqrt{2}\,\frac{m+n}{n}\right)^n = \sum c_\nu^{(4)} z^\nu. \tag{6.5}$$

The degree of f_4 is

$$\leqq \frac{n}{2}\left(3+\left[\frac{\pi}{\varkappa}\right]\right)-1. \tag{6.6}$$

Further the inequality

$$c_\nu^{(4)} \geqq 0 \tag{6.7}$$

holds for all ν's and let us observe that owing to (6.2) and the nonnegativity of the $c_\nu^{(3)}$ coefficients for $\nu = 0, 1, \dots, n-1$ even the inequality

$$c_\nu^{(4)} \geqq 2\left(4\sqrt{e}\,\frac{m+n}{n}\right)^n \tag{6.8}$$

holds.

7. The last two auxiliary polynomials with real coefficients will be

$$f_5(z) = f_4(z) + f_1(z) = \sum_\nu c_\nu^{(5)} z^\nu \tag{7.1}$$

$$f_6(z) = f_4(z) - f_1(z) = \sum_\nu c_\nu^{(6)} z^\nu. \tag{7.2}$$

Owing to (6.6) their degree cannot exceed

$$\frac{n}{2}\left(3+\left[\frac{\pi}{\varkappa}\right]\right)-1. \tag{7.3}$$

We assert further that for all ν's

$$c_\nu^{(5)} \geqq 0, \quad c_\nu^{(6)} \geqq 0; \tag{7.4}$$

it will be enough to show the first ones. Since $f_1(z)$ has a degree $\leqq n-1$, (6.7) clearly implies $c_\nu^{(5)} \geqq 0$ for $\nu \geqq n$. But also for $\nu \leqq n-1$ the assertion is true owing to (6.8) and (5.6). Replacing further z by z_j ($j = 1, \dots, n$) we get

$$\sum_\nu c_\nu^{(5)} z_j^\nu = z_j^{-m-1},$$

owing to (5.1) and (6.5), (6.2), (6.1), i.e.

$$\sum_\nu c_\nu^{(5)} z_j^{m+1+\nu} = 1, \quad j = 1, \dots, n. \tag{7.5}$$

Quite analogously we get

$$\sum_\nu c_\nu^{(6)} z_j^{m+1+\nu} = -1, \quad j = 1, \dots, n.$$

Multiplying then by b_j we get

$$\begin{aligned} \sum_\nu c_\nu^{(5)} g(m+\nu+1) &= g(0) \\ \sum_\nu c_\nu^{(6)} g(m+\nu+1) &= -g(0). \end{aligned} \tag{7.6}$$

Owing to (7.3) we have

$$m+1 \leqq m+1+\nu \leqq m+\frac{n}{2}\left(3+\left[\frac{\pi}{\varkappa}\right]\right);$$

since owing to (2.7)—(2.8) the $g(m+\nu+1)$'s are all real and we may suppose without loss of generality

$$g(0) = |g(0)|,$$

we get from (7.6) the inequalities

$$\max g(\nu) \geqq \frac{|g(0)|}{\sum c_\nu^{(5)}} \tag{7.7}$$

$$\min g(\nu) \leqq -\frac{|g(0)|}{\sum c_\nu^{(6)}},$$

where the max resp. min refers to the integers of the interval

$$m+1 \leqq \nu \leqq m+\frac{n}{2}\left(3+\left[\frac{\pi}{\varkappa}\right]\right). \tag{7.8}$$

8. In order to complete the proof of Theorem 2 we have to give upper bounds for $\sum c_\nu^{(5)}$ and $\sum c_\nu^{(6)}$; it is enough to do it for the first one. [7.1], [6.5] and Lema 3 give

$$\sum c_\nu^{(5)} = f_4(1)+f_1(1) \leqq f_4(1)+\sum_\nu |c_\nu^{(1)}| \leqq \tag{8.1}$$

$$\leqq f_3(1)2n\left(4\sqrt{e}\,\frac{m+n}{n}\right)^n+\left(4\sqrt{e}\,\frac{m+n}{n}\right)^n.$$

Using finally (6.2)—(6.4) we get

$$\sum c_\nu^{(5)} \leqq \left(8\sqrt{e}\,\frac{m+n}{n}\right)^n(2n+1)$$

which completes the proof.

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