On the commutativity of non-associative rings

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1. Many sufficient conditions are well known under which a given ring becomes commutative. Notable among them are some given by JACOBSON, KAPLANSKY and HERSTEIN. In all these results, they take the ring to be associative. Recently JOHNSEN, OUTCALT and YAQUB [3] proved that if R is a non-associative ring with unity, in which $(xy)^2=x^2y^2$ holds for all x, y in R, then R is commutative. Further, GUPTA [2] proved that if R is a non-associative ring with unity satisfying the condition $(xy)^2=(yx)^2$ for all x, y in R and additive group of R has no element of order 2, then R is commutative. In this paper our object is to generalize the above mentioned results.

2. Throughout the paper the rings considered are not necessarily associative, but contains unity. A ring R is said to be n-torsion free for any positive integer n, if whenever for any x in R, $nx=0$ implies that $x=0$. For any $x_1, x_2, \ldots, x_n$ in R, we define $x_1\cdot x_2\cdots x_n=(x_1\cdot x_2\cdots x_{n-1})x_n$.

Now we begin with the following lemma which we need throughout this paper

Lemma 1. *For any positive integers n and m, the following relations hold:*

(i) $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m+1-k)^n=0$ *or* $n!$ *according as* $m>$ *or* $=n$,

(ii) *For any positive integers n and t, $n\geqq 2$ and $1\leqq t\leqq n-1$*

$$\sum_{r=0}^{n-t-1}\left\{\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t+r-1}\right\}\binom{n}{t+r}=\frac{t+1}{t}\sum_{k=0}^{t}(-1)^k\binom{t}{k}(t+1-k)^{n-1}.$$

PROOF. Relation (i) follows from exercise 1 of [1, p. 154]. *Left hand side* of (ii) is equal to

$$\sum_{k=0}^{t-1}\left\{\sum_{r=0}^{n-t-1}\binom{n}{t+r}(t-k)^{t+r-1}\right\}(-1)^k\binom{t-1}{k}$$

or,

$$\sum_{k=0}^{t-1}\frac{1}{t-k}\left[\left\{(1+(t-k))^n-(t-k)^n-1\right\}-\sum_{p=1}^{t-1}\binom{n}{p}(t-k)^p\right](-1)^k\binom{t-1}{k}$$

or,

$$\sum_{k=0}^{t-1} \frac{1}{t-k} \left\{(1+(t-k))^n-(t-k)^n-1\right\}(-1)^k \binom{t-1}{k} -$$
$$- \sum_{k=0}^{t-1} \sum_{p=1}^{t-1} \binom{n}{p} (t-k)^{p-1}(-1)^k \binom{t-1}{k}$$

or,

$$\frac{t+1}{t} \sum_{k=0}^{t} \binom{t}{k} (-1)^k (t+1-k)^{n-1} - \sum_{p=1}^{t-1} \binom{n}{p} \left\{\sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} (t-k)^{p-1}\right\}.$$

In view of relation (i) and the last equation we arrive at the conclusion.

3. Rings in which $(xy)^n=(yx)^n$.

Lemma 2. *For a fixed positive integer n, let R satisfy $(xy)^n=(yx)^n$ for all $x, y \in R$ and let $(R, +)$ be a p-torsion free for each prime $p<n$, then for all x, y in R*

$$(y^{n-1} \cdot xy + y^{n-2} \cdot xy \cdot y + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-1)\text{-times}}) = (y^{n-1} \cdot yx + \ldots + yx \cdot \overbrace{y \cdots y}^{(n-1)\text{-times}}).$$

Proof. By hypothesis $(xy)^n = (yx)^n$ and $((x+1)y)^n = (y(x+1))^n$ holds for all x, y in R. Therefore $((x+1)y)^n - (xy)^n = (y(x+1))^n - (yx)^n$. Then

$$\begin{aligned}
(1) \qquad & \{(y \cdot \overbrace{xy \cdot xy \cdots xy}^{(n-1)\text{-times}} + xy \cdot y \cdot \overbrace{xy \cdots xy}^{(n-2)\text{-times}} + \ldots + \overbrace{xy \cdot xy \cdots xy}^{(n-1)\text{-times}} \cdot y) \\
& + (y^2 \cdot \overbrace{xy \cdots xy}^{(n-2)\text{-times}} + y \cdot xy \cdot y \cdot xy \cdots xy + \ldots + \overbrace{xy \cdots xy}^{(n-2)\text{-times}} \cdot y \cdot y) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + (y^{n-1} \cdot xy + y^{n-2} \cdot xy \cdot y + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-1)\text{-times}})\} = A_1
\end{aligned}$$

where A_1 is obtained from left hand side of (1) by replacing xy by yx in each term and keeping all other factors unchanged. On both sides of (1) number of terms in respective brackets are $\binom{n}{r}$ $(r=1, 2, \ldots, n-1)$.

In (1) replace x by $x+1$ and use (1). Then, we get

$$\begin{aligned}
(2) \qquad & \{2(2-1)(y^2 \cdot \overbrace{xy \cdots xy}^{(n-2)\text{-times}} + y \cdot xy \cdot y \cdot xy \cdots xy + \ldots + \overbrace{xy \cdots xy}^{(n-2)\text{-times}} \cdot y \cdot y) \\
& + 2(2^2-1)(y^3 \cdot xy \cdots xy + y^2 \cdot xy \cdot y \cdot xy \cdots xy + \ldots + xy \cdots xy \cdot y \cdot y \cdot y) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + 2(2^{r-1}-1)(y^r \cdot \overbrace{xy \cdots xy}^{(n-r)\text{-times}} + y^{r-1} \cdot xy \cdot y \cdot xy \cdots xy + \ldots + \overbrace{xy \cdots xy}^{(n-r)\text{-times}} \cdot \overbrace{y \cdots y}^{r\text{-times}}) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + 2(2^{n-2}-1)(y^{n-1} \cdot xy + y^{n-2} \cdot xy \cdot y + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-2)\text{-times}})\} = A_2
\end{aligned}$$

where A_2 is obtained from the L. H. S. of (2) by replacing in each term xy by yx and all other factors unchanged. The terms within the r-th bracket of the L. H. S. of (2) are same as those in $(r+1)$th bracket in L. H. S. of (1).

Further the coefficient $2(2^{r-1}-1)$ outside the r-th bracket of (2) is obtained as follows:

Consider $y^r \cdot \overbrace{xy \cdots xy}^{(n-r)\text{-timet}}$, take any term within the first bracket of (1), and delet x from $r-1$ places in that term, we obtain a term involving xy, $n-r$ times and y, r times. Since we want to obtain $y^r \cdot xy \cdots xy$, we take only those terms in the first bracket of (1) which involves xy, $n-r$ times at the end. From any such term, if we delete x, $r-1$ times we obtain $y^r \cdot xy \cdots xy$. There are $\binom{r}{r-1}$ such terms in the first bracket of (1). Analogous procedure, when applied to kth bracket $(k=1, \ldots, r-1)$ of L. H. S. of (1) gives $y^r \cdot xy \cdots xy$, $\binom{r}{r-k}$ times. No term in any bracket after $(r-1)$th bracket give $y^r \cdot xy \cdots xy$. Thus we obtain $\binom{r}{r-1}+\ldots+\binom{r}{1}=2(2^{r-1}-1)$ as the coefficient of $y^r \cdot xy \cdots xy$. By symmetry all other terms in $(r-1)$th bracket of (2) have the same coefficient, namely $2(2^{r-1}-1)$.

To apply the induction, suppose by continuing the same process as above for t-times, we obtain

$$B_t + B_{t+1} + \ldots + B_{n-1} = A_t \tag{3}$$

where for $r = t, t+1, \ldots, n-1$,

$$B_r = t \cdot \sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} (t-k)^{r-1} (y^r \cdot \overbrace{xy \cdots xy}^{(n-r)\text{-times}} + \ldots + \overbrace{xy \cdots xy}^{(n-r)\text{-times}} \cdot \overbrace{y \cdots y}^{r\text{-times}}$$

where A_t is obtained from the L. H. S. of (3) by replacing xy by yx in each term, and keeping their factors unchanged. In (3) replace x by $x+1$ and use (3), we prove that we get the following:

$$C_{t+1} + C_{t+2} + \ldots + C_{n-1} = A_{t+1} \tag{4}$$

where for $r = t+1, \ldots, n-1$,

$$C_r = (t+1) \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t+1-k)^{r-1} \cdot (y^r \cdot \overbrace{xy \cdots xy}^{(n-r)\text{-times}} + \ldots + \overbrace{xy \cdots xy}^{(n-r)\text{-times}} \cdot \overbrace{y \cdot y \cdots y}^{r\text{-times}}),$$

and A_{t+1} is obtained from the left hand side of (4) by replacing in each term xy by yx and keeping other factors unchanged.

The common coefficient outside the $(l-t)$th bracket, where $t+1 \leqq l \leqq n-1$ of (4) obtained by the same argument as for (2) and is equal to

$$t \cdot \sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} (t-k)^{t-1} \binom{l}{t-1} + \ldots + t \cdot \sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} (t-k)^{l-2} \binom{l}{1}$$

or,

$(t+1) \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t+1-k)^{l-1}$ by Lemma 1 (ii). This proves the assertion.

Putting $t=n-1$ in (3) we get

$$(n-1)\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-1-k)^{n-2}(y^{n-1}\cdot xy+\ldots+xy\cdot\overbrace{y\cdots y}^{(n-1)\text{-times}})=$$

$$=(n-1)\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-1-k)^{n-2}(y^{n-1}\cdot yx+\ldots+yx\cdot\overbrace{y\cdots y}^{(n-1)\text{-times}}).$$

In view of Lemma 1 (i) the last equation reduces to

$$(n-1)!\,(y^{n-1}\cdot xy+\ldots+xy\cdot y\cdots y)=(n-1)!\,(y^{n-1}\cdot yx+\ldots+yx\cdot y\cdots y).$$

Since $(R,+)$ is p-torsion free for each prime integer $p<n$,

$$(y^{n-1}\cdot xy+\ldots+xy\overbrace{\cdot y\cdots y}^{(n-1)\text{-times}})=(y^{n-1}\cdot yx+\ldots+yx\overbrace{\cdot y\cdots y}^{(n-1)\text{-times}}).$$

Theorem 1. *Let R be a ring (not necessarily associative) with unity $1\neq 0$, such that $(xy)^n=(yx)^n$ for some fixed positive integer $n\geqq 1$, and for all x, y in R. Further let the additive group of R be p-torsion free for every prime integer $p\leqq n$. Then R is commutative.*

PROOF. By Lemma 2, for all x, y in R

$$(y^{n-1}\cdot xy+\ldots+xy\overbrace{\cdot y\cdots y}^{(n-1)\text{-times}})=(y^{n-1}\cdot yx+\ldots+yx\overbrace{\cdot y\cdots y}^{(n-1)\text{-times}}).\tag{5}$$

Replace y by $y+1$ in (5) and use (5). After simple computation and rearranging terms we obtain

$$\begin{aligned}&\Big\{\binom{n}{1}(y^{n-2}\cdot xy+x+\ldots+xy\cdot\overbrace{y\cdots y}^{(n-2)\text{-times}})\\&+\binom{n}{2}(y^{n-3}\cdot xy+\ldots+xy\cdot y\cdots y)\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+\binom{n}{r-1}(y^{n-r}\cdot xy+\ldots+xy\cdot\overbrace{y\cdots y}^{(n-r)\text{-times}})\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+\binom{n}{n-1}(y^0\cdot xy)\Big\}=B_1\end{aligned}\tag{6}$$

where B_1 is obtained from the L. H. S. of (6) by replacing xy by yx in each term and keeping other factors unchanged. The number of terms in respective brackets are $(n-r)$ for $r=1,2,\ldots,n-1$.

To apply induction, suppose by continuing the same process as above for t-times, we obtain

$$C_t+C_{t+1}+\ldots+C_{n-1}=A_t\tag{7}$$

where for $r=t, t+1, \dots, n-1$,

$$C_r = t\binom{n}{r}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{r-1}(y^{n-r-1}\cdot xy+\dots+xy\cdot\overbrace{y\cdots y}^{(n-r-1)\text{-times}}),$$

and A_t is obtained by replacing xy by yx in each term of L. H. S. of (7) and keeping remaining factors unchanged. The number of terms in respective brackets of L. H. S. of (7) are $n-r\,(r=t, t+1, \dots, n-1)$.

By replacing y by $y+1$ in (7) and using (7), we prove that we get

$$D_{t+1}+D_{t+2}+\dots+D_{n-1} = A_{t+1} \tag{8}$$

where for $r=t+1, \dots, n-1$,

$$D_r = (t+1)\binom{n}{r}\sum_{k=0}^{t}(-1)^k\binom{t}{k}(t+1-k)^{r-1}(y^{n-r-1}\cdot xy+\dots+xy\cdot\overbrace{y\cdots y}^{(n-r-1)\text{-times}})$$

and A_{t+1} is obtained from L. H. S. of (8) by replacing xy by yx in each term and keeping other factors unchanged.

The coefficient outside the $(l-t-1)$th bracket of L. H. S. of (8) is obtained as follows, where $l=t+2, \dots, n$.

The contribution to the coefficient of $y^{n-l}\cdot xy$ $(t+2\leqq l\leqq n)$ from the resulting expression obtained after putting $y+1$ in place of y in (7), from the first bracket of (7), is

$$t\binom{n}{t}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t-1}\binom{n-t}{l-t-1},$$

where we have used the identity

$$\binom{n-l}{0}+\binom{n-l+1}{1}+\dots+\binom{n-1-t}{l-1-t} = \binom{n-t}{l-1-t},$$

a direct consequence of the well-known identity

$$\sum_{\gamma=0}^{h}\binom{m+\gamma}{\gamma} = \binom{m+h+1}{h} \quad (m \text{ and } h \text{ positive integers}).$$

Similarly by the same process as above, the contribution to the coefficient of $y^{x-l}\cdot xy$ from the resulting expression obtained after putting $y+1$ in place of y in (7), from the $(l-t-1)$th bracket of (7) is

$$t\binom{n}{l-2}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-3}\binom{n-l+2}{1},$$

where we have used the identity

$$\binom{n-l}{0}+\binom{n-l+1}{1} = \binom{n-l+2}{1}.$$

The contribution to the coefficient of $y^{n-l}\cdot xy$ after the $(l-t-1)$th bracket is zero. Therefore, the total contribution to the coefficient of $y^{n-l}\cdot xy$ is

$$\Big\{t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t-1}\binom{n}{t}\binom{n-t}{l-t-1}$$

$$+t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t}\binom{n}{t+1}\binom{n-t-1}{l-t-2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-3}\binom{n}{l-2}\binom{n-l+2}{1}\Big\}.$$

Since,

$$\binom{n}{t}\binom{n-t}{l-t-1}=\binom{n}{l-1}\binom{l-1}{t},\ \ldots,\ \binom{n}{l-2}\binom{n-l+2}{1}=\binom{n}{l-1}\binom{l-1}{l-2}.$$

Therefore, the above expression becomes

$$t\binom{n}{l-1}\Big\{\sum_{k=0}^{t}(-1)^k\binom{t-1}{k}(t-k)^{t-1}\binom{l-1}{t}+\ldots+\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-3}\binom{l-1}{l-2}\Big\}.$$

In view of Lemma 1 (ii), the last expression becomes

$$(t+1)\binom{n}{l-1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}(t+1-k)^{l-2}\quad\text{for all}\quad l=t+2,\ldots,n-1.$$

This proves the assertion. Putting $t=n-1$ in (7). Then

$$(n-1)\binom{n}{n-1}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-k-1)^{n-2}(xy)=$$

$$=(n-1)\binom{n}{n-1}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-k-1)^{n-2}(yx).$$

By Lemma 1 (i), we get $(n)!xy=(n)!yx$ for all x, y in R. Since additive group of R is torsion free for each prime integer $p\leqq n$. We get $xy=yx$ for all $x, y\in R$. Hence R is commutative.

4. Rings in which $(xy)^n=x^ny^n$.

Lemma 3. *If R is a ring such that $(xy)^n=x^ny^n$ holds for some positive fixed integer $n>1$ and for all x and y in R, and moreover the additive group of R is p-torsion free for each prime integer $p<n$, then $(y^{n-1}\cdot xy+\ldots+xy\cdot\overbrace{y\cdots y}^{(n-1)\text{-times}})=n\cdot xy^n$ holds for all $x, y\in R$.*

PROOF. Since for all $x, y \in R$, $(xy)^n = x^n y^n$ and $((x+1)y)^n = (x+1)^n y^n$ holds. On subtraction, we get

$$\begin{aligned}
(9) \qquad & \{(y \cdot x\overbrace{y \cdots xy}^{(n-1)\text{-times}} + \ldots + x\overbrace{y \cdot xy \cdots xy}^{(n-1)\text{-times}} \cdot y \\
& + (y^2 + xy \cdots xy + \ldots + xy \cdots xy \cdot y \cdot y) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + (y^{n-1} \cdot xy + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-1)\text{-times}})\} \\
& = \binom{n}{1} x^{n-1} y^n + \binom{n}{2} x^{n-2} y^n + \ldots + \binom{n}{n-1} xy^n.
\end{aligned}$$

The number of terms in the respective brackets on both sides of (9) are $\binom{n}{r}$ $(r = 1, 2, \ldots, n)$.

To apply induction, suppose by continuing the same process as above for t-time, we get

$$(10) \qquad D_t + D_{t+1} + \ldots + D_{n-1} = E_t + E_{t+1} + \ldots + E_{n-1}$$

where for $r = t, t+1, \ldots, n-1$,

$$D_r = t \cdot \sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} (t-k)^{r-1} (y^r \cdot \overbrace{xy \cdots xy}^{(n-r)\text{-times}} + \ldots + \overbrace{xy \cdots xy}^{(n-r)\text{-times}} \cdot \overbrace{y \cdots y}^{r\text{-times}})$$

and

$$E_r = t \cdot \binom{n}{r} \cdot \sum_{k=0}^{t-1} (-1)^k \binom{t-1}{k} (t-k)^{r-1} (x^{n-r} \cdot y^n).$$

Replacing x by $x+1$ in (10) as using (10), we claim that we get

$$(11) \qquad F_{t+1} + \ldots + F_{n-1} = G_{t+1} + \ldots + G_{n-1}$$

where for $r = t+1, \ldots, n-1$,

$$F_r = (t+1) \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t+1-k)^{r-1} (y^r \cdot \overbrace{xy \cdots xy}^{(n-r)\text{-times}} + \ldots + \overbrace{xy \cdots xy}^{(n-r)\text{-times}} \cdot \overbrace{y \cdots y}^{r\text{-times}})$$

and

$$G_r = (t+1) \binom{n}{r} \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t+1-k)^{r-1} (x^{n-r} y^n).$$

The coefficient, $(t+1) \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t+1-k)^{l-1}$ outside the $(l-t)$th bracket $(t+1 \leqq l \leqq n-1)$ in L. H. S. of (11) is obtained as the same way as in Lemma 2. And the coefficinte $(t+1) \binom{n}{l} \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t+1-k)^{l-1}$ outside the $(l-t)$th bracket $(t+1 \leqq l \leqq n-1)$ in the R. H. S. of (11) is obtained as follows:

The contribution to the coefficient of $x^{n-l}\cdot y^n$ from the first bracket of the resulting expression obtained by putting $x+1$ in place of x in (10), is

$$t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t-1}\binom{n}{t}\binom{n-t}{l-t}.$$

The contribution to the coefficient of $x^{n-l}\cdot y^n$ from the $(l-t)$th bracket of the resulting expression obtained by putting $x+1$ in place of x in R. H. S. of (10), is

$$t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-2}\binom{n}{l-1}\binom{n-l+1}{1}.$$

The contribution to the coefficient of $x^{n-l}\cdot y^n$ after the $(l-t)$th bracket of (10) is zero. Thus the total contribution to the coefficient of $x^{n-l}\cdot y^n$ from (10) is

$$\begin{aligned}\Bigg\{&t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t-1}\binom{n}{t}\binom{n-t}{l-t}\\ &+t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t}\binom{n}{t+1}\binom{n-t-1}{l-t-1}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &+t\cdot\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-2}\binom{n}{l-1}\binom{n-l+1}{1}\Bigg\}.\end{aligned}$$

Since

$$\binom{n}{t}\binom{n-t}{l-t}=\binom{n}{l}\binom{l}{t},\ \dots,\ \binom{n}{l-1}\binom{n-l+1}{1}=\binom{n}{l}\binom{l}{l-1}.$$

The above expression becomes

$$t\cdot\binom{n}{l}\cdot\left\{\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{t-1}\binom{l}{t}+\dots+\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-2}\binom{l}{l-1}\right\}.$$

In view of Lemma 1 (ii), the last equation yields

$$(t+1)\binom{n}{l}\sum_{k=0}^{t}(-1)^k\binom{t}{k}(t+1-k)^{l-1}$$

which proves the claim. Putting $t=n-1$ in (10) we get

$$\begin{aligned}&(n-1)\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-k-1)^{n-2}(y^{n-1}\cdot xy+\dots+xy\cdot\overbrace{y\cdot y\cdots y}^{(n-1)\text{-times}})\\ &\quad=(n-1)\cdot n\cdot\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-k-1)^{n-2}(xy^n).\end{aligned}$$

By Lemma 1 (i) we have $(n-1)!(y^{n-1}\cdot xy+\dots+xy\cdot y\cdots y)=(n-1)!(n\cdot xy^n)$. Since additive group of R is p-torsion free for every prime integer $p<n$,

$$(y^{n-1}\cdot xy+\dots+xy\cdot\overbrace{y\cdots y}^{(n-1)\text{-times}})=n\cdot xy^n.$$

Theorem 2. *Let R be any non-associative ring with $1 \neq 0$ such that for some fixed integer $n > 1$, $(xy)^n = x^n y^n$ holds for all x, y in R, and the additive group of R is p-torsion free for every prime integer $p \leqq n$, then R is commutative; if n is even then it is sufficient to take $p < n$.*

PROOF. By Lemma 3, $(y^{n-1}xy + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-1)\text{-times}}) = n \cdot xy^n$ holds for all x, y in R. Replace y by $y+1$ in the last equation and subtract the same equation from the resulting expression. Then

$$\left[\left\{\binom{n}{1}(y^{n-2} \cdot xy + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-2)\text{-times}}) + \binom{n}{0}(y^{n-1} \cdot x + y^{n-2} \cdot x \cdot y + \ldots + x \cdot \overbrace{y \cdot y \cdots y}^{(n-1)\text{-times}})\right\}\right.$$

$$+\left\{\binom{n}{2}(y^{n-3} \cdot xy + y^{n-4} \cdot xy \cdot y + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-3)\text{-times}}) + \right.$$

$$\left. + \binom{n}{1}(y^{n-2} \cdot x + y^{n-3} \cdot x \cdot y + \ldots + x \cdot \overbrace{y \cdot y \cdots y}^{(n-2)\text{-times}})\right\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\left\{\binom{n}{n-2}(y \cdot xy + xy \cdot y) + \binom{n}{n-3}(y^2 \cdot x + y \cdot x \cdot y + x \cdot y \cdot y)\right\} +$$

$$\left. + \left\{\binom{n}{n-1}(xy) + \binom{n}{n-2}(yx + xy)\right\}\right]$$

$$= n\left\{\binom{n}{1} xy^{n-1} + \binom{n}{2} xy^{n-2} + \ldots + \binom{n}{r} x \cdot y^{n-r} + \ldots + \binom{n}{n-1} xy\right\}.$$

To apply the induction, suppose by continuing the same process as above for $(t-1)$-times, we obtain

$$D_t + D_{t+1} + \ldots + D_n = E_t + E_{t+1} + \ldots + E_n \tag{12}$$

where for $r = t, t+1, \ldots, n$,

$$D_r = (t-1)\left\{\binom{n}{r-1}\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-k-1)^{r-2}(y^{n-r} \cdot xy + \ldots + xy \cdot \overbrace{y \cdots y}^{(n-r)\text{-times}})\right.$$

$$\left. + \binom{n}{r-2}\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-k-1)^{r-2}(y^{n-r+1} \cdot x + \ldots + x \cdot \overbrace{y \cdot y \cdots y}^{(n-r+1)\text{-times}})\right\}$$

and

$$E_r = n \cdot (t-1) \cdot \binom{n}{r-1}\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-k-1)^{r-2}(xy^{n-r+1}).$$

Replace y by $(y+1)$ in (12) and subtract (12) from the resulting expression. We prove that we get

$$F_{t+1} + F_{t+2} + \ldots + F_n = G_{t+1} + G_{t+2} + \ldots + G_n \tag{13}$$

where for $r=t+1, t+2, \ldots, n$,

$$F_r = t\left\{\binom{n}{r-1}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{r-2}(y^{n-r}\cdot xy+\ldots+xy\cdot\overbrace{y\cdots y}^{(n-r)\text{-times}})\right.$$
$$\left.+\binom{n}{r-2}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{r-2}(y^{n-r+1}\cdot x+\ldots+x\cdot\overbrace{y\cdot y\cdots y}^{(n-r+1)\text{-times}})\right\}$$

and

$$G_r = n\cdot t\binom{n}{r-1}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{r-2}(xy^{n-r+1}),$$

The coefficient, $n\cdot t\cdot\binom{n}{l-1}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-2}$ of xy^{n-l+1} $(t+1\leqq l\leqq n)$ in the R. H. S. of (13) is obtained in the same way as in Lemma 3. The coefficient $t\binom{n}{l-1}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-2}$ of $(y^{n-l}\cdot xy+\ldots+xy\cdot\overbrace{y\cdots y}^{(n-l)\text{-times}})$, for $l=t+1, \ldots, n$, of the L. H. S. of (13) is obtained in the same way as in Theorem 1. And the coefficient $t\binom{n}{l-2}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-2}$ of $(y^{n-l+1}\cdot x+\ldots+x\cdot\overbrace{y\cdot y\cdots y}^{(n-l+1)\text{-times}})$ for $l=$ $=t+1, \ldots, n$ in the L. H. S. of (13) is obtained as follows:

The contribution to the coefficient of $y^{n-l+1}\cdot x$, $(t+1\leqq l\leqq n)$ from the first term of the first bracket of the resulting expression obtained by putting $y+1$ in place of y in (12) is,

$$(t-1)\binom{n}{t-1}\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-1-k)^{t-2}\binom{n-t+1}{l-t-1},$$

where we have used the identity

$$\binom{n-l+1}{0}+\binom{n-l+2}{1}+\ldots+\binom{n-t}{l-t-1}=\binom{n-t-1}{l-t-1},$$

and the contribution to the coefficient of $y^{n-l+1}\cdot x$ from the second term of the first bracket of the resulting expression obtained by putting $y+1$ in place of y in (12) is

$$(t-1)\binom{n}{t-2}\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-1-k)^{t-2}\binom{n-t+2}{l-t},$$

where we have used the identity

$$\binom{n-l+1}{0}+\binom{n-l+2}{1}+\ldots+\binom{n-t+1}{l-t}=\binom{n-t+2}{l-t}.$$

Hence the total contribution to the coefficient of $y^{n-l+1}\cdot x$ from the first bracket of the resulting expression obtained after putting $y+1$ in place of y in (12), is

$$(t-1)\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-k-1)^{t-2}\binom{n}{l-2}\binom{l-1}{t-1},$$

because,

$$\binom{n}{t-1}\binom{n-t+1}{l-t-1}+\binom{n}{t-2}\binom{n-t+2}{l-t}=\binom{n}{l-2}\binom{l-2}{t-1}+\binom{n}{l-2}\binom{l-2}{t-2}=$$
$$=\binom{n}{l-2}\binom{l-1}{t-1}.$$

Similarly as above, the contribution to the coefficient of $y^{n-l+1}\cdot x$ from the $(l-t)$th bracket of the resulting expression obtained by putting $y+1$ in place of y in (12) is,

$$(t-1)\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-1-k)^{l-3}\binom{n}{l-2}\binom{l-2}{l-1}.$$

The contribution to the coefficient of $y^{n-l+1}\cdot x$ after the $(l-t)$th bracket is zero. Therefore the total contribution to the coefficient of $y^{n-l+1}\cdot x$ from all brackets of (12) is,

$$\begin{aligned}(t-1)\binom{n}{l-2}\Bigg\{&\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-k-1)^{t-2}\binom{l-1}{t-1}\\ &+\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-k-1)^{t-1}\binom{l-1}{t}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &+\sum_{k=0}^{t-2}(-1)^k\binom{t-2}{k}(t-k-1)^{l-3}\binom{l-1}{l-2}\Bigg\}.\end{aligned}$$

By Lemma 1 (ii) the last equation gives

$$t\binom{n}{l-2}\sum_{k=0}^{t-1}(-1)^k\binom{t-1}{k}(t-k)^{l-2}.$$

This proves the assertion, putting $t=n$ in (12) we get

$$\begin{aligned}&(n-1)\Bigg\{n\cdot\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-k-1)^{n-2}\cdot(xy)\\ &+\binom{n}{n-2}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-k-1)^{n-2}(xy+yx)\Bigg\}=\\ &=n\cdot(n-1)\cdot n\cdot\sum_{k=0}^{n-2}\binom{n-2}{k}(-1)^k(n-k-1)^{n-2}(xy).\end{aligned}$$

In view of Lemma 1 (i), we get

$$(n-1)\cdot n\cdot(n-2)!\,(xy)+(n-1)\cdot\binom{n}{2}\cdot(n-2)!\,(yx+xy)=n\cdot(n-1)\cdot n\cdot(n-2)!\,(xy)$$

or,

$$(n-1)!\binom{n}{2}(xy)=(n-1)!\binom{n}{2}(yx).$$

Since the additive group of R is p-torsion free for each prime integer $p \leqq n$, we obtain $xy = yx$ for all $x, y \in R$. If n is even, $n/2$ divides $(n-1)!$, so in assumption it is sufficient to take $p < n$.

5. By the same process as we proved above theorems, we can prove

Theorem 3. *If R is a non-associative ring with 1 and satisfies for fixed $n \geqq 1$, $(xy)^n = (y^n x^n)$ for each $x, y \in R$ and the additive group of R is p-torsion free for each prime $p \leqq n+1$. Then R is commutative. If n is odd it is sufficient to take $p \leqq n$.*

Theorem 4. *If R is a non-associative ring with 1 satisfying for a fixed $n \geqq 1$, $x^n y^n = y^n x^n$ for each $x, y \in R$ and the additive group of R is p-torsion free for each prime $p \leqq n$. Then R is commutative.*

6. *Remark.* Example (3) of Johnsen, Outcalt and Yaqub [3] show that there exists a ring R with 1 and satisfying the polynomial identity in above theorems and the additive group of R is p-torsion, where p divides n if n is odd and p divides $n/2$ if n is even. Morevoer, R is not commutative.

It remains an open question that the assumption of $(R, +)$ is p-torsion free for each prime $p < n$ and $(p, n) = 1$ in above theorems is essential or not.

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