Analytic solutions of a system of functional equations

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This paper deals with the problem of the existence, uniqueness and continuous dependence on the given functions of the local analytic solutions of the system of functional equations

$$\varphi_i(z) = h_i(z; \varphi_1[f_1(z)], \dots, \varphi_1[f_n(z)]; \dots; \varphi_m[f_1(z)], \dots, \varphi_m[f_n(z)]), \quad i = 1, \dots, m,$$

where h_i, $i=1, \dots, m$, and f_k, $k=1, \dots, n$, are given functions defined and analytic in a neighbourhood of the point $(0, \dots, 0) \in C^{nm+1}$ and $0 \in C$, repectively. C stands here and in the sequel for the field of all complex numbers.

The problem of the existence and uniqueness of analytic solutions of systems of functional equations has been a subject of [8] and [9]. In the case of single equations this problem has been investigated in [7] (cf. also [5]) and [10]. In the latter case the papers [2]—[4] deal with the problem of the continuous dependence of analytic solutions on the given functions.

Before proceeding with the statement of our result we shall give some notations and definitions.

For the shake of simplicity we shall write

$$(z; w_{j,k})$$

instead of

$$(z; w_{1,1}, \dots, w_{1,n}; \dots; w_{m,1}, \dots, w_{m,n})$$

for every complex numbers z and $w_{j,k}$, assuming (unless explicity stated otherwise) that the indexes i and j run over the set $\{1, \dots, m\}$, k runs over $\{1, \dots, n\}$, while v runs over the set of all nonnegative integers. Moreover, we put

$$C(r) := \{z \in C : |z| \leqq r\}$$

for a positive number r, and

$$a_{i,j} := \sum_{k=1}^{n} a_{i,j,k}, \tag{1}$$

$$a^1_{\lambda,\mu} := \begin{cases} a_{\lambda,\mu} & \text{for } \lambda \neq \mu \\ 1 - a_{\lambda,\mu} & \text{for } \lambda = \mu \end{cases}; \quad \lambda, \mu = 1, \dots, m,$$

$$a^{\varkappa+1}_{\lambda,\mu} := \begin{cases} a^{\varkappa}_{1,1} a^{\varkappa}_{\lambda+1,\mu+1} + a^{\varkappa}_{\lambda+1,1} a^{\varkappa}_{1,\mu+1} & \text{for } \lambda \neq \mu \\ a^{\varkappa}_{1,1} a^{\varkappa}_{\lambda+1,\mu+1} - a^{\varkappa}_{\lambda+1,1} a^{\varkappa}_{1,\mu+1} & \text{for } \lambda = \mu \end{cases}; \tag{2}$$

$$\varkappa = 1, \dots, m-1; \ \lambda, \mu = 1, \dots, m-\varkappa,$$

for arbitrary numbers $a_{i,j,k}$.

Using these notations we shall consider the sequence of systems of functional equations

$$\varphi_i(z) = h_{i,\nu}\big(z;\varphi_j[f_{k,\nu}(z)]\big) \tag{3}$$

under the following hypotheses regarding the given functions:

(i) $f_{k,\nu}$ are analytic in $C(r)$ for a positive r and $|f_{k,\nu}(z)|\leqq|z|$ in $C(r)$.

(ii) $h_{i,\nu}$ are analytic in $C(r)\times C^{nm}$ and $h_{i,\nu}(0,\ldots,0)=0$;

(iii) the sequences $\{f_{k,\nu}\}_{\nu=1}^{\infty}$ and $\{h_{i,\nu}\}_{\nu=1}^{\infty}$ tend to $f_{k,0}$ and $h_{i,0}$ uniformly in $C(r)$ and in every compact subset of $C(r)\times C^{nm}$, respectively.

We define the functions $h_{i,\nu,\sigma}$, $\sigma=1,2,\ldots$, by the recurrent relations

$$h_{i,\nu,1}(z;w_{j,k};v_{j,k}^1):=\frac{\partial h_{i,\nu}}{\partial z}(z;w_{j,k})+\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial h_{i,\nu}}{\partial w_{j,k}}(z;w_{j,k})f'_{k,\nu}(z)v_{j,k}^1,$$
$$(z;w_{j,k};v_{j,k}^1)\in C(r)\times C^{nm}\times C^{nm},$$

$$\begin{aligned} h_{i,\nu,\sigma+1}(z;w_{j,k};v_{j,k}^1;\ldots;v_{j,k}^{\sigma+1}) &:= \frac{\partial h_{i,\nu,\sigma}}{\partial z}(z;w_{j,k};v_{j,k}^1;\ldots;v_{j,k}^{\sigma}) \\ &+\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial h_{i,\nu,\sigma}}{\partial w_{j,k}}(z;w_{j,k};v_{j,k}^1;\ldots;v_{j,k}^{\sigma})f'_{k,\nu}(z)v_{j,k}^1 \\ &+\sum_{s=1}^{\sigma}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\partial h_{i,\nu,\sigma}}{\partial v_{j,k}^s}(z;w_{j,k};v_{j,k}^1;\ldots;v_{j,k}^{\sigma})f'_{k,\nu}(z)v_{j,k}^{s+1}, \end{aligned} \tag{4}$$
$$(z;w_{j,k};v_{j,k}^1;\ldots;v_{j,k}^{\sigma})\in C(r)\times C^{nm(\sigma+1)},\quad \sigma=1,2,\ldots.$$

By the induction principle one can easily derive the following

Lemma. *Suppose that hypotheses* (i) *and* (ii) *are satisfied. If φ_i are analytic in $C(r)$ and*

$$\psi_{i,\nu}(z):=h_{i,\nu}\big(z;\varphi_j[f_{k,\nu}(z)]\big),\quad z\in C(r),$$

then

$$\psi_{i,\nu}^{(\sigma)}(z)=h_{i,\nu,\sigma}\big(z;\varphi_j[f_{k,\nu}(z)];\varphi_j^{(1)}[f_{k,\nu}(z)];\ldots;\varphi_j^{(\sigma)}[f_{k,\nu}(z)]\big),$$
$$\sigma=1,2,\ldots,z\in C(r),$$

where $h_{i,\nu,\sigma}$, $\sigma=1,2,\ldots$, are defined by (4).

As a direct consequence of (i), (ii) and the preceding lemma we get that the derivatives

$$\varphi_{i,\nu}^{(\sigma)}(0)=:c_{i,\nu,\sigma},\quad \sigma=1,\ldots,\sigma_0 \tag{5}$$

of the functions $\varphi_{i,\nu}$ being a solution of (3) such that $\varphi_{i,\nu}(0)=0$, must satisfy the system

$$\begin{aligned} c_{i,\nu,\sigma}=h_{i,\nu,\sigma}(0;\underbrace{0,\ldots\,0}_{nm};\underbrace{c_{1,\nu,1},\ldots,c_{1,\nu,1}}_{n},\ldots,\underbrace{c_{m,\nu,1},\ldots,c_{m,\nu,1}}_{n};\ldots \\ \ldots;\underbrace{c_{1,\nu,\sigma},\ldots,c_{1,\nu,\sigma}}_{n},\ldots,\underbrace{c_{m,\nu,\sigma},\ldots,c_{m,\nu,\sigma}}_{n}), \end{aligned} \tag{6}$$
$$\sigma=1,\ldots,\sigma_0,$$

for an arbitrary positive integer σ_0.

Theorem. *Suppose that hypotheses* (i)—(iii) *are satisfied. If there exists a positive integer σ_0 such that the inequalities*

$$a^{\varkappa}_{\lambda,\lambda,\nu} > 0, \quad \varkappa = 1, \dots, m, \ \lambda = 1, \dots, m+1-\varkappa, \tag{7}$$

are satisfied, where $a^{\varkappa}_{\lambda,\mu,\nu}$, $\varkappa=1, \dots, m$, $\lambda, \mu=1, \dots, m+1-\varkappa$, are obtained as the result of operations (1) *and* (2) *applied to the numbers*

$$a_{i,j,k,\nu} := \left| \frac{\partial h_{i,\nu}}{\partial w_{j,k}}(0, \dots, 0)[f'_{k,\nu}(0)]^{\sigma_0} \right|, \tag{8}$$

then there exists an open neighbourhood U of the origin such that for every ν and for every system of numbers $c_{i,\nu,\sigma}$, $\sigma=1, \dots, \sigma_0$, fulfilling (6) *system* (3) *has exactly one analytic solution $\varphi_{i,\nu}$ in U for which the equalities $\varphi_{i,\nu}(0)=0$ and* (5) *hold. Moreover, $\{\varphi_{i,\nu}\}_{\nu=1}^{\infty}$ tends to $\varphi_{i,0}$ uniformly in U, whenever*

$$\lim_{\nu\to\infty} c_{i,\nu,\sigma} = c_{i,0,\sigma}, \quad \sigma = 1, \dots, \sigma_0. \tag{9}$$

PROOF. Let us define the polynomials

$$P_{i,\nu}(z) := \sum_{\sigma=1}^{\sigma_0} \frac{1}{\sigma!} c_{i,\nu,\sigma} z^{\sigma} \tag{10}$$

with $c_{i,\nu,\sigma}$, $\sigma=1, \dots, \sigma_0$, being a given solution of (6) and put

$$H_{i,\nu}(z; w_{j,k}) := \frac{1}{z^{\sigma_0}} \left[h_{i,\nu}\left(z; P_{j,\nu}[f_{k,\nu}(z)] + [f_{k,\nu}(z)]^{\sigma_0} w_{j,k}\right) - P_{i,\nu}(z)\right], \tag{11}$$

$$z \in C(r) \setminus \{0\}, \ w_{j,k} \in C.$$

We shall prove that $H_{i,\nu}$ are analytic in $C(r) \times C^{nm}$. Write

$$g_{i,\nu}(z) := h_{i,\nu}\left(z; P_{j,\nu}[f_{k,\nu}(z)]\right) - P_{i,\nu}(z), \quad z \in C(r).$$

It follows from hypotheses (i), (ii) and from (6) and (10) that these functions are analytic, $g_{i,\nu}(0)=0$ and $g^{(\sigma)}_{i,\nu}(0)=0$ (see lemma) for $\sigma=1, \dots, \sigma_0$. Hence $H_{i,\nu}(\cdot\,; 0)$ are analytic in $C(r)$ and

$$H_{i,\nu}(0, \dots, 0) = 0. \tag{12}$$

Since

$$\frac{\partial H_{i,\nu}}{\partial w_{j,k}}(z; w_{j,k}) = \left[\frac{f_{k,\nu}(z)}{z}\right]^{\sigma_0} \frac{\partial h_{i,\nu}}{\partial w_{j,k}}\left(z; P_{j,\nu}[f_{k,\nu}(z)] + [f_{k,\nu}(z)]^{\sigma_0} w_{j,k}\right),$$

$$z \in C(r) \setminus \{0\}, \ w_{j,k} \in C,$$

we infer in view of (i) and (ii) that $\frac{\partial H_{i,\nu}}{\partial w_{j,k}}$ are analytic in $C(r) \times C^{nm}$. Bearing these facts in mind in virtue of the relation

$$H_{i,\nu}(z; w_{1,1}, 0, \dots, 0) = \int_0^{w_{1,1}} \frac{\partial H_{i,\nu}}{\partial w}(z; w, 0, \dots, 0)\, dw + H_{i,\nu}(z; 0, \dots, 0),$$

$$z \in C(r) \setminus \{0\}, \ w_{11,} \in C,$$

we get the analyticity of $H_{i,\nu}(\cdot\,;\,\cdot,0,\dots,0)$ in $C(r)\times C$. Inductively, one can obtain the analyticity of $H_{i,\nu}$ in $C(r)\times C^{nm}$. Moreover,

(13) $$H_{i,\nu}\xrightarrow[\nu\to\infty]{} H_{i,0},\ \text{uniformly in every compact subset of } C(r)\times C^{nm},$$

in view of (iii).

Making use of (7) one may easily verify that there exists an $\varepsilon>0$ such that the inequalities

(14) $$b^{\varkappa}_{\lambda,\lambda,\nu}>0,\quad \varkappa=1,\dots,m,\ \lambda=1,\dots,m+1-\varkappa,$$

are satisfied with $b^{\varkappa}_{\lambda,\mu,\nu}$, $\varkappa=1,\dots,m$, $\lambda,\mu=1,\dots,m+1-\varkappa$, obtained as the result of operations (1) and (2) applied to the numbers

(15) $$b_{i,j,k,\nu}:=a_{i,j,k,\nu}+\varepsilon.$$

Recalling (8) and (11), for every ν, we are able to choose positive numbers $r_{0,\nu}\leqq r$ such that

(16) $$|H_{i,\nu}(z;w_{j,k})-H_{i,\nu}(z;\hat{w}_{j,k})|\leqq\sum_{j=1}^{m}\sum_{k=1}^{n}\left(a_{i,j,k,\nu}+\frac{\varepsilon}{2}\right)|w_{j,k}-\hat{w}_{j,k}|,$$
$$(z;w_{j,k}),\ (z;w_{j,k})\in C(r_{0,\nu})^{nm+1}.$$

Hence by (13) and (15) there exists a positive integer ν_0 such that

(17) $$|H_{i,\nu}(z;w_{j,k})-H_{i,\nu}(z;\hat{w}_{j,k})|\leqq\sum_{j=1}^{m}\sum_{k=1}^{n}b_{i,j,k,0}|w_{j,k}-\hat{w}_{j,k}|,$$
$$\nu\geqq\nu_0,\quad (z;w_{j,k}),(z;\hat{w}_{j,k})\in C(r_{0,0})^{nm+1}.$$

Put
$$r_0:=\min\{r_{0,0},\dots,r_{0,\nu_0}\}.$$

Then by (15), (16) and (17) we get

(18) $$|H_{i,\nu}(z;w_{j,k})-H_{i,\nu}(z;\hat{w}_{j,k})|\leqq\begin{cases}\sum_{j=1}^{m}\sum_{k=1}^{n}b_{i,j,k,\nu}|w_{j,k}-\hat{w}_{j,k}|, & \nu=0,\dots,\nu_0-1,\\ \sum_{j=1}^{m}\sum_{k=1}^{n}b_{i,j,k,0}|w_{j,k}-\hat{w}_{j,k}|, & \nu\geqq\nu_0,\end{cases}$$
$$(z;w_{j,k}),(z;\hat{w}_{j,k})\in C(r_0)^{nm+1}.$$

By (14) one may apply the lemma from [6] which allows to find positive numbers $R_{i,\nu}$ such that $R_{i,\nu}\leqq r_0$ and

(19) $$\sum_{j=1}^{m}\left(\sum_{k=1}^{n}b_{i,j,k,\nu}\right)R_{j,\nu}<R_{i,\nu}.$$

Putting $z=\hat{w}_{j,k}=0$ in (18) and using (12) we obtain

$$|H_{i,\nu}(0;\, w_{j,k})| \leqq \begin{cases} \sum_{j=1}^{m} \sum_{k=1}^{n} b_{i,j,k,\nu} |w_{j,k}|, & \nu = 0, \dots, \nu_0 - 1, \\ \sum_{j=1}^{m} \sum_{k=1}^{n} b_{i,j,k,0} |w_{j,k}|, & \nu \geqq \nu_0, \end{cases}$$

$$w_{j,k} \in C(r_0).$$

Thus in view of (19)

$$|H_{i,\nu}(0;\, w_{j,k})| < \begin{cases} R_{i,\nu} & \text{for} \quad w_{j,k} \in C(R_{j,\nu}), \quad \nu = 0, \dots, \nu_0 - 1, \\ R_{i,0} & \text{for} \quad w_{j,k} \in C(R_{j,0}), \quad \nu \geqq \nu_0. \end{cases}$$

Hence, there exists an r_1, $0 < r_1 \leqq r_0$, such that

$$(20) \qquad |H_{i,\nu}(z;\, w_{j,k})| \leqq \begin{cases} R_{i,\nu} & \text{for} \quad (z;\, w_{j,k}) \in C(r_1) \times C(R_{j,\nu})^{nm}, \quad \nu = 0, \dots, \nu_0 - 1, \\ R_{i,0} & \text{for} \quad (z;\, w_{j,k}) \in C(r_1) \times C(R_{j,0})^{nm}, \quad \nu \geqq \nu_0. \end{cases}$$

Now, we denote by $A_{i,\nu}$, $\nu = 0, \dots, \nu_0 - 1$, the complete metric space of all functions Φ defined and continuous in $C(r_1)$, analytic in $U := \operatorname{int} C(r_1)$ and such that $\Phi(0) = 0$, $|\Phi(z)| \leqq R_{i,\nu}$ for $z \in C(r_1)$, $\nu = 0, \dots, \nu_0 - 1$, endowed with the supremum metric ϱ. Let

$$B_{i,\nu} := \begin{cases} A_{i,\nu} & \text{for} \quad \nu = 0, \dots, \nu_0 - 1 \\ A_{i,0} & \text{for} \quad \nu \geqq \nu_0 \end{cases}$$

and

$$T_{i,\nu}(\Phi_1, \dots, \Phi_m)(z) := H_{i,\nu}(z;\, \Phi_j[f_{k,\nu}(z)]), \quad \Phi_i \in B_{i,\nu}, \quad z \in C(r_1).$$

From (i), (ii), (12) and (20) we get the inclusions

$$(21) \qquad T_{i,\nu}(B_{1,\nu} \times \dots \times B_{m,\nu}) \subset B_{i,\nu}.$$

Furthermore, in view of (18)

$$(22) \qquad \varrho\big(T_{i,\nu}(\Phi_1, \dots, \Phi_m),\, T_{i,\nu}(\hat{\Phi}_1, \dots, \hat{\Phi}_m)\big) \leqq$$

$$\leqq \begin{cases} \sum_{j=1}^{m} \left(\sum_{k=1}^{n} b_{i,j,k,\nu} \right) \varrho(\Phi_j, \hat{\Phi}_j), & \nu = 0, \dots, \nu_0 - 1, \\ \sum_{j=1}^{m} \left(\sum_{k=1}^{n} b_{i,j,k,0} \right) \varrho(\Phi_j, \hat{\Phi}_j), & \nu \geqq \nu_0, \end{cases} \qquad \Phi_j,\, \hat{\Phi}_j \in B_{j,\nu}.$$

The relationships (21), (22) and (14) allow us to apply the fixed-point theorem from [6]. Thus, for every ν, the system

$$(23) \qquad \Phi_i(z) = H_{i,\nu}\big(z;\, \Phi_j[f_{k,\nu}(z)]\big)$$

has exactly one solution $\Phi_{i,\nu} \in B_{i,\nu}$. Therefore, the functions $\varphi_{i,\nu}$ defined by

$$(24) \qquad \varphi_{i,\nu}(z) := P_{i,\nu}(z) + z^{\sigma_0} \Phi_{i,\nu}(z), \quad z \in U,$$

where $P_{i,\nu}$ are given by (10), yield an analytic solution of (3) in U.

It follows from our lemma that if $\varphi_{i,\nu}$ yield an analytic solution of (3) in U such that $\varphi_{i,\nu}(0)=0$ and $\varphi_{i,\nu}^{(\sigma)}(0)=c_{i,\nu,\sigma}$, $\sigma=1, \dots, \sigma_0$, where $c_{i,\nu,\sigma}$, $\sigma=1, \dots, \sigma_0$, fulfil (6), then they must be of the form (24) with $P_{i,\nu}$ given by (10). Recalling definition (11) of $H_{i,\nu}$ and representation (24) we see that $\Phi_{i,\nu}$ fulfil (23) in U as well as the condition $\Phi_{i,\nu}(0)=0$. This shows that for every ν and for every system of numbers $c_{i,\nu,\sigma}$, $\sigma=1, \dots, \sigma_0$, fulfilling (6) system (3) has exactly one analytic solution $\varphi_{i,\nu}$ such that $\varphi_{i,\nu}(0)=0$ and $\varphi_{i,\nu}^{(\sigma)}(0)=c_{i,\nu,\sigma}$, $\sigma=1, \dots, \sigma_0$. These solutions are defined in a common neighbourhood U of the origin.

We shall show that $\{\varphi_{i,\nu}\}_{\nu=1}^{\infty}$ tends to $\varphi_{i,0}$ uniformly in U, whenever (9) holds. Evidently, in view of (9), (10) and (24) it suffices to show that $\{\Phi_{i,\nu}\}_{\nu=1}^{\infty}$ tends to $\Phi_{i,0}$ uniformly in $C(r_1)$. It follows from (18), (13) and hypothesis (iii) that

$$(25) \qquad \lim_{\nu\to\infty} T_{i,\nu}(\Phi_1, \dots, \Phi_m) = T_{i,0}(\Phi_1, \dots, \Phi_m), \quad \Phi_j \in A_{j,0}.$$

In fact,

$$\begin{aligned} &\varrho\big(T_{i,\nu}(\Phi_1, \dots, \Phi_m), T_{i,0}(\Phi_1, \dots, \Phi_m)\big) = \\ &= \sup\left\{\left|H_{i,\nu}\big(z;\, \Phi_j[f_{k,\nu}(z)]\big) - H_{i,0}\big(z;\, \Phi_j[f_{k,0}(z)]\big)\right| : z\in C(r_1)\right\} \\ &\leqq \sup\left\{\left|H_{i,\nu}\big(z;\, \Phi_j[f_{k,\nu}(z)]\big) - H_{i,\nu}\big(z;\, \Phi_j[f_{k,0}(z)]\big)\right| : z\in C(r_1)\right\} \\ &+ \sup\left\{\left|H_{i,\nu}\big(z;\, \Phi_j[f_{k,0}(z)]\big) - H_{i,0}\big(z;\, \Phi_j[f_{k,0}(z)]\big)\right| : z\in C(r_1)\right\}, \end{aligned}$$

for every $\nu\geqq\nu_0$ and $\Phi_j\in A_{j,0}$, which proves (25). Moreover,

$$\Phi_{i,\nu} = T_{i,\nu}(\Phi_{1,\nu}, \dots, \Phi_{m,\nu}).$$

Thus in view of (21), (22), (14) and (25) (cf. lemma in [1])

$$\Phi_{i,\nu} \xrightarrow[\nu\to\infty]{} \Phi_{i,0}, \quad \text{uniformly in} \quad C(r_1),$$

which completes the proof of our theorem.

References

[1] K. Baron, A few observations regarding continuous solutions of a system of functional equations, *Publ. Math. Debrecen* **21** (1974), 185—191.

[2] J. Matkowski, On the continuous dependence of solutions of a functional equation on given functions, *Ann. Polon. Math.* **23** (1970), 37—40.

[3] ———, On the continuous dependence of local analytic solutions of a functional equation on given functions, *Ann. Polon. Math.* **24** (1970), 21—26.

[4] ———, On the continuous dependence of local analytic solutions of the functional equation in the non-uniqueness case, *Ann. Polon. Math.* **24** (1971), 319—326.

[5] ———, Note on a functional equation, *Zeszyty Nauk. Jagiell. Uniw. Jagiell. Prace Matematyczne* **15** (1971), 109—111.

[6] ———, Some inequalities and a generalization of Banach's principle, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **21** (1973), 323—324.

[7] W. Smajdor, On the existence and uniqueness of analytic solutions of the functional equation $\varphi(z)=h(z, \varphi[f(z)])$, *Ann. Polon. Math.* **19** (1967), 37—45.

[8] ———, Local analytic solutions of the functional equation $\varphi(z)=h(z, \varphi[f(z)])$ in multidimensional spaces, *Aequationes Math.* **1** (1968), 20—36.

[9] ———, Analytic solutions of the equation $\varphi(z)=h(z, \varphi[f(z)])$ with right side contracting, *Aequationes Math.* **2** (1968), 30—38.

[10] ———, Local analytic solutions of the functional equation $\Phi(z)=H(z;\, \Phi[f_1(z)], \dots, \Phi[f_m(z)])$, *Ann. Polon. Math.* **24** (1970), 39—43.

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