Sum theorems for the covering dimension of totally normal spaces

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A sum theorem for the covering dimension of normal spaces has been proved in [1]. Also, in [5], K. NAGAMI obtains a sum theorem for the covering dimension of paracompact Hausdorff spaces. Making use of these sum theorems, we establish in the present note several sum theorems for the covering dimension of totally normal spaces as well as for that of hereditarily paracompact spaces. All spaces are assumed to be normal and T_1.

1. *Sum theorems for the covering dimension of totally normal spaces*

We first give the definition of the covering dimension of a space.

A subset Y of a space X is said to be a separating set if $X \sim Y$ is not connected. If H and K are disjoint subsets of X, and Y is a separating subset of X such that there exists a relatively open-closed subset G of $X \sim Y$ with $H \subseteqq G$ and $G \cap K = \emptyset$, then Y is said to separate H and K.

The covering dimension of X, denoted by $\dim X$, is $\leqq n$ if every finite open covering of X can be refined by an open covering whose order is at most $n+1$. If $\dim X \leqq n$ and the statement $\dim X \leqq n-1$ is false, we say that $\dim X = n$. If the statement $\dim X \leqq n$ is false for all n, we say that $\dim X = \infty$. Also, $\dim \emptyset = -1$.

We shall need the following theorems proved in [1] and in [5].

Theorem 1.1 [ENGELKING, 1]. *If X is a normal space and $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed covering of X such that* $\dim F_\alpha \leqq n$ *for all* $\alpha \in \Lambda$, *then* $\dim X \leqq n$.

Theorem 1.2 [NAGAMI, 5]. *If X is a totally normal space and $Y \subseteqq X$, then* $\dim Y \leqq \dim X$.

A family $\{A_\alpha : \alpha \in \Lambda\}$ is said to be order locally finite [2] if there is a linear ordering $<'$ of the index set Λ such that for each $\alpha \in \Lambda$, the family $\{A_\beta : \beta < \alpha\}$ is locally finite at each point of A_α.

Obviously, every σ-locally finite family is order locally finite but not conversely.

Theorem 1.3. *Let $\mathscr{V}$ be an locally order finite open covering of a totally normal space X such that* $\dim \overline{V} \leqq n$ *for each* $V \in \mathscr{V}$. *Then* $\dim X \leqq n$.

Proof. Let $\mathscr{V}=\{V_\alpha: \alpha\in\Lambda\}$ where Λ is a linearly ordered index set such that $\{V_\beta: \beta<\alpha\}$ is locally finite at each point of V_α for each $\alpha\in\Lambda$. Let $\mathscr{F}=\{F_\alpha: \alpha\in\Lambda\}$ where $F_\alpha=\overline{V}_\alpha\sim\bigcup\{V_\beta: \beta<\alpha\}$ for each $\alpha\in\Lambda$. In view of theorem 1.2 above, $\dim F_\alpha\leqq n$ for each $\alpha\in\Lambda$. Also, since $\mathscr{V}$ is order locally finite, it is easy to see that $\mathscr{F}$ is locally finite. Thus $\mathscr{F}$ is a locally finite closed covering of X such that $\dim F_\alpha\leqq n$ for each $\alpha\in\Lambda$. Hence $\dim X\leqq n$ in view of theorem 1.1 above.

Corollary 1.1. If $\mathscr{V}$ is a σ-locally finite open covering of a totally normal space X such that $\dim \overline{V}\leqq n$ for each $V\in\mathscr{V}$, then $\dim X\leqq n$.

Corollary 1.2. Let $J=\{F_\alpha: \alpha\in\Lambda\}$ be an order locally finite family of closed subsets of a totally normal space X such that $U\{F_\alpha^\circ: \alpha\in\Lambda\}=X_\lambda$. If $\dim F_\alpha\leqq n$ for lach $\alpha\in\Lambda$, than $\dim X\leqq n$.

Proof. $\{F_\alpha^0: \alpha\in\Lambda\}$ is an locally order finite open covering of the totally normal space X such that $\dim F_\alpha^{0-}\leqq n$ for each $\alpha\in\Lambda$, since $F_\alpha^{0-}\subseteqq F_\alpha$ and $\dim F_\alpha\leqq n$ for each α. It follows that $\dim X$ is $\leqq n$ view of theorem 1.3 above.

Theorem 1.4. *Let $\mathscr{V}$ be an order locally finite open covering of a totally normal space X such that* $\dim V\leqq n$ *for each $V\in\mathscr{V}$. If frontier V is compact for each $V\in\mathscr{V}$, then* $\dim X\leqq n$.

Proof. Let $V\in\mathscr{V}$ be fixed. Let $H=\overline{V}\sim V$. Since H is compact and X is regular, there exist finitely many open sets $\{G_i: i=1, 2, \dots, p\}$ covering H such that each $\overline{G}_i\subseteqq V^*$ for some $V^*\in\mathscr{V}$. For each $i=1, \dots, p$, let $F_i=\overline{G}_i\cap\overline{V}$ and let $F_0=\overline{V}\sim\bigcup_{i=1}^{p} G_i$. Then $\mathscr{F}=\{F_i: i=0, 1, \dots, p\}$ is a finite (and hence locally finite) closed covering of $\overline{V}$ such that $\dim F_i\leqq n$ for each i. Hence $\dim\overline{V}\leqq n$ in view of theorem 1.1 and hence $\dim X\leqq n$ in view of theorem 1.3.

Corollary 1.3. If $\mathscr{V}$ is a σ-locally finite open covering of a totally normal space X such that $\dim V\leqq n$ for each $V\in\mathscr{V}$ and frontier V is compact for each $V\in\mathscr{V}$, then $\dim X\leqq n$.

Theorem 1.5. *If $\{G_\alpha: \alpha\in\Lambda\}$ is a locally finite open covering of a totally normal space X such that* $\dim G_\alpha\leqq n$ *for each $\alpha\in\Lambda$, then* $\dim X\leqq n$.

Proof. Since X is normal, there exists a locally finite open covering $\{V_\alpha: \alpha\in\Lambda\}$ of X such that $\overline{V}_\alpha\subseteqq G_\alpha$ for each $\alpha\in\Lambda$. Now, $\dim\overline{V}_\alpha\leqq n$ for each $\alpha\in\Lambda$ in view of theorem 1.2. Thus, $\{\overline{V}_\alpha: \alpha\in\Lambda\}$ is a locally finite closed covering of X such that $\dim\overline{V}_\alpha\leqq n$ for each $\alpha\in\Lambda$. Hence $\dim X\leqq n$ in view of theorem 1.1.

Theorem 1.6. *Let $\{G_\alpha: \alpha\in\Lambda\}$ be a normal open covering of a totally normal space X. If* $\dim G_\alpha\leqq n$ *for each $\alpha\in\Lambda$, then* $\dim X\leqq n$.

Proof. $\{G_\alpha: \alpha\in\Lambda\}$ is a normal covering, therefore in view of theorem 1.2 in [3], $\{G_\alpha: \alpha\in\Lambda\}$ has a locally finite open refinement $\{V_\beta: \beta\in\Delta\}$. Now, proceeding as in the proof of theorem 1.5 above, it can easily be proved that $\dim X\leqq n$.

Added in Prof. The authors have recently shown that if $\{G_\alpha: \alpha\in\Lambda\}$ be any open covering of a hereditarily paracompact Hausdorff space X such that $\dim G_\alpha\leqq n$ for all $\alpha\in\Lambda$ (or $\dim\overline{G}_\alpha\leqq n$ for all $\alpha\in\Lambda$), then dim $X\leqq n$. All theorems in sections 2 follow as corollaries of this result.

Corollary 1.4. Let $\{G_\alpha: \alpha\in\Lambda\}$ be a σ-locally finite open covering of a countably paracompact, totally normal space X such that $\dim G_\alpha \leqq n$ for each $\alpha\in\Lambda$. Then $\dim X\leqq n$.

Proof. Every σ-locally finite open covering of a countably paracompact, normal space is a normal covering [4].

Corollary 1.5. Let $\{G_\alpha: \alpha\in\Lambda\}$ be a σ-locally finite open covering of a totally normal space X such that each G_α is an F_σ-subset of X. If $\dim G_\alpha\leqq n$ for each $\alpha\in\Lambda$, then $\dim X\leqq n$.

Proof. $\{G_\alpha: \alpha\in\Lambda\}$ is a normal covering in view of theorem 1.2 in [4].

2. *Sum theorems for the covering dimension of hereditarily paracompact spaces*

In this section, we shall obtain some sum theorems for the covering dimension of hereditarily paracompact spaces. We shall need the following theorem proved in [5].

Theorem 2.1 [Nagami, 5]. *If* $\{F_\alpha: \alpha\in\Lambda\}$ *is a locally countable closed covering of a paracompact space X such that* $\dim F_\alpha\leqq n$ *for each* $\alpha\in\Lambda$, *then* $\dim X\leqq n$.

Every hereditarily paracompact space is totally normal and obviously paracompact. (As a matter of fact, every paracompact, totally normal space is hereditarily paracompact.) Since we wish to make use of both theorems 1.2 and 2.1, we shall consider spaces which are hereditarily paracompact.

Generalizing the notion of order locally finite families we introduce the notion of order locally countable families.

A family $\{A_\alpha: \alpha\in\Lambda\}$ is said to be order locally countable if there exists a linear ordering "$<$" of the index set Λ such that for each $\alpha\in\Lambda$, the family $\{A_\beta: \beta<\alpha\}$ is locally countable at each point of A_α.

Obviously, every σ-locally countable family is order locally countable but not conversely.

Theorem 2.2. *Let* $\mathcal{G}$ *be an order locally countable open covering of a hereditarily paracompact space X such that* $\dim \overline{G}\leqq n$ *for each* $G\in\mathcal{G}$. *Then* $\dim X\leqq n$.

Proof. Let $\mathcal{G}=\{G_\alpha: \alpha\in\Lambda\}$ where Λ is a linearly ordered index set such that $\{G_\beta: \beta<\alpha\}$ is locally countable at each point of G_α for each $\alpha\in\Lambda$. For each α, let $A_\alpha=\overline{G}_\alpha\sim\bigcup\{G_\beta: \beta<\alpha\}$. Since $A_\alpha\subseteq\overline{G}_\alpha$ and $\dim\overline{G}_\alpha\leqq n$, therefore, $\dim A_\alpha\leqq n$ for each $\alpha\in\Lambda$ in view of theorem 1.2. Obviously, $\{A_\alpha: \alpha\in\Lambda\}$ is a closed covering of X. We shall prove that $\{A_\alpha: \alpha\in\Lambda\}$ is locally countable. Let $x\in X$. Since $\mathcal{G}$ is a covering of X, there exists $\alpha\in\Lambda$ such that $x\in G_\alpha$. Consider $\{G_\beta: \beta<\alpha\}$. $\mathcal{G}$ being order locally countable and x being a point of G_α, there exists an open set U such that $x\in U$ and U intersects at most countably many members of $\{G_\beta: \beta<\alpha\}$ and hence also at most countably many members of $\{A_\beta: \beta<\alpha\}$. Also, in view of the construction, $G_\alpha\cap A_\beta=\emptyset$ for all $\beta>\alpha$. Thus $U\cap G_\alpha$ is a neighbourhood of x which intersects at most countably many members of $\{A_\alpha: \alpha\in\Lambda\}$. $\{A_\alpha: \alpha\in\Lambda\}$ is thus a locally countable closed covering of the paracompact space X such that $\dim A_\alpha\leqq n$ for each $\alpha\in\Lambda$. Hence $\dim X\leqq n$ in view of theorem 2.1.

Corollary 2.1. If $\mathcal{G}$ is a countable open covering of a hereditarily paracompact space X such that $\dim \overline{G} \leqq n$ for each $G \in \mathcal{G}$, then $\dim X \leqq n$.

Corollary 2.2. If $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ is a order locally countable family of closed subsets of a hereditarily paracompact space X such that $\cup \{A_\alpha^0 : \alpha \in \Lambda\} = X$ and if $\dim A_\alpha \leqq n$ for each $\alpha \in \Lambda$, then $\dim X \leqq n$.

Theorem 2.3. *If $\mathcal{G}$ is an order locally countable open covering of a hereditarily paracompact space X such that frontier G is compact for each $G \in \mathcal{G}$ and* $\dim G \leqq n$ *for each $G \in \mathcal{G}$, then* $\dim X \leqq n$.

PROOF. Proceeding as in the proof of theorem 1.4, it can be proved that there exists a finite and hence a locally countable closed covering of $\overline{G}$ for each $G \in \mathcal{G}$ such that the covering dimension of each member of the covering is $\leqq n$. Hence $\dim G \leqq n$ for each $G \in \mathcal{G}$. It follows that $\dim X \leqq n$ in view of theorem 2.2 above.

Corollary 2.3. If $\mathcal{G}$ is a σ-locally countable open covering of a hereditarily paracompact space X such that frontier G is compact for each $G \in \mathcal{G}$ and $\dim G \leqq n$ for each $G \in \mathcal{G}$, then $\dim X \leqq n$.

Theorem 2.4. *Let $\mathcal{G}$ be a σ-locally countable open covering of a hereditarily paracompact space X such that each $G \in \mathcal{G}$ is an F_σ-set and* $\dim G \leqq n$ *for each $G \in \mathcal{G}$. Then* $\dim X \leqq n$.

PROOF. Let $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ where each $\mathcal{G}_i$ is locally countable. Let $\mathcal{G}_i = \{G_\alpha : \alpha \in \Lambda_i\}$. Since X is normal and each G_α is an F_σ-set, therefore for each $\alpha \in \Lambda_i$, $G_\alpha = \bigcup_{i=1}^{\infty} \mathcal{G}_{\alpha, i}$, where each $G_{\alpha, i}$ is an open set whose closure is contained in G_α. Let $\mathcal{H}_{i,j} = \{G_{\alpha, j} : \alpha \in \Lambda_i\}$ and let $\mathcal{H} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{H}_{i,j}$. Then $\mathcal{H}$ is a σ-locally countable open covering of X such that $\dim \overline{H} \leqq n$ for each $H \in \mathcal{H}$. Hence $\dim X \leqq n$ in view of corollary 2.1.

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