Contributions to the theory of semimodular lattices

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1. *Introduction.* It is well-known that semimodularity was firstly introduced only for lattices of finite length (G. BIRKHOFF, [1], p. 445). Birkhoff's definition, based on covering property of elements, was not suitable of course to be extended directly for lattices of infinite length[1]). Some years later R. CROISOT gave a definition of semimodularity in general, coinciding with the Birkhoff's one in the case of finite length. His definition was formulated in [2] as follows:

Definition 1. A lattice L is called *semimodular* if and only if to each triplet $a, b, x \in L$ with the properties[2])

$$a \| b \quad \text{and} \quad a \frown b < x < a$$

there exists an element y such that

$$(1) \qquad a \frown b < y \leqq b \quad \text{and} \quad (x \smile y) \frown a = x.$$

One sees at once that the elements a and b do not take symmetric parts in this definition, although they do in Birkhoff's definition. But, as we show in Section 2 of this note, Croisot's definition can be rewritten into a symmetric form. After having proved the equivalence of the new definition to the original one we add some remarks to this definition.

The subject of Section 3 is related to § 4 of [4]. Theorem 6 of that paper shows that there exist complemented semimodular lattices in which no inner element has a maximal or a minimal complement[3]). Now we deal with lattices satisfying the lower covering condition and we prove a theorem on relative complements from which we can derive, as a special case, the following counterpole of the result quoted just now: In a complemented semimodular lattice each element of finite height has both maximal and minimal complements. In addition, we show that every element of a partition lattice has the same property.

2. *On the definition of semimodularity.* We rephrase the definition of semimodularity as follows:

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[1]) For notations and terminology used without being explained in this note, we refer to [3].

[2]) $a \| b$ means that the elements a and b are incomparable.

[3]) We call the attention of the reader to the fact that the lattice discussed in that theorem is not relatively complemented in general.

Definition 2. A lattice L is called semimodular if and only if to each triplet $a, b, x \in L$ with the properties

$$a \| b \quad \text{and} \quad a \frown b < x \leqq a$$

there exists an element z such that

$$a \frown b < z \leqq b, \tag{2}$$

$$(x \smile z) \frown a = x, \tag{3}$$

$$(x \smile z) \frown b = z. \tag{4}$$

In order to legitimate this definition we have to prove

Theorem 1. *Definitions 1 and 2 are equivalent over the class of lattices.*

Proof. Let C_i $(i=1, 2)$ denote the class of lattices that are semimodular in the sense of Definition i. Then $C_2 \subseteqq C_1$ obviously. Thus we have to verify the reversed inclusion.

Let L be any lattice from the class C_1 and let a, b be any pair of incomparable elements of L. Further, let x denote any element of L such that $a \frown b < x \leqq a$.

In case of $x = a$ we make the choice $z = b$ which meets the requirements (2)—(4) trivially.

In case of $x < a$ there exists an $y \in L$ satisfying (1) by Definition 1. Choose

$$z = (x \smile y) \frown b. \tag{5}$$

Then

$$b \geqq z \geqq y \frown b = y > a \frown b,$$

verifying the inequalities in (2), and

$$x \smile y \geqq z \geqq y,$$

implying

$$x \smile y \geqq x \smile z \geqq x \smile y,$$

whence

$$x \smile y = x \smile z.$$

From this equation we get (3) and (4) by direct calculation:

$$(x \smile z) \frown a = (x \smile y) \frown a = x$$

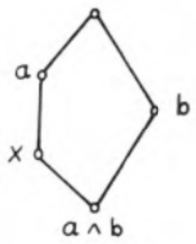
by (1), and

$$(x \smile z) \frown b = (x \smile y) \frown b = z$$

by (5). Hence, each lattice L contained in the class C_1 belongs to C_2. Thus, Theorem 1 is proved.

Remark 1. In the French version of [3] we have pointed out that in case of a modular lattice L the requirements in Definition 1 can be satisfied by choosing $y = b$. Actually, *a lattice is modular if and only if* (3) *can be solved by* $z = b$ *for each* x

satisfying the inequalities $a \frown b < x \leqq a$; in fact, any non-modular lattice has a sublattice S represented by the diagram



in which $(x \smile b) \frown a = a \neq x$.

Remark 2. Moreover, if L is modular, then (3) and (4) are valid with *any* x and z satisfying the inequalities

$$a \frown b \leqq x \leqq a \quad \text{and} \quad a \frown b \leqq z \leqq b,$$

respectively. In fact, these inequalities imply the equations

$$a \frown z = a \frown b = x \frown b$$

whereby, using the modularity, we get

$$x = x \smile (a \frown b) = x \smile (z \frown a) = (x \smile z) \frown a,$$

$$z = z \smile (a \frown b) = z \smile (x \frown b) = (z \smile x) \frown b.$$

3. *Existence of minimal and maximal complements.* For any elements u, v, a with $u \leqq a \leqq v$ let $R_u^v(a)$ denote the set of the relative complements of a in $[u, v]$. We prove the following

Theorem 2. *Let* $[u, v]$ *be an interval, satisfying the lower covering condition, of a lattice L. If a is an element of* $[u, v]$ *such that the length of interval* $[u, a]$ *is finite, then the length of the partly ordered set* $R_u^v(a)$ *is not greater than that of* $[u, a]$.

Corollary. Every element of finite height of a complemented semimodular lattice has maximal as well as minimal complements.

Proof. Since the theorem is clearly true for the cases $a = u$ and $a = v$, we may restrict the discussion to the case when $u < a < v$.

Provided that the length of $[u, a]$ is n (where n is finite), there exists a chain[4])

$$u = a_0 \prec a_1 \prec \ldots \prec a_{n-1} \prec a_n = a \tag{6}$$

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[4]) $x \prec y$ means that x is covered by y.

between u and a. Let r denote any relative complement of a in $[u, v]$:

$$a \frown r = u, \quad a \smile r = v.$$

Form the chain

$$r = a_0 \smile r \leqq a_1 \smile r \leqq \ldots \leqq a_{n-1} \smile r \leqq a_n \smile r = v. \tag{7}$$

We show that it is a maximal chain between r and v. Since

$$a_j \leqq a_{j+1} \frown (a_j \smile r) \leqq a_{j+1} \quad (j = 0, 1, \ldots, n-1)$$

and $a_j \prec a_{j+1}$ by (6), either

$$a_j = a_{j+1} \frown (a_j \smile r) \prec a_{j+1}$$

or

$$a_{j+1} \frown (a_j \smile r) = a_{j+1}.$$

In the former case

$$a_j \smile r \prec a_{j+1} \smile (a_j \smile r) = a_{j+1} \smile r$$

by the lower covering condition. In the latter case $a_j \leqq a_{j+1} \leqq a_j \smile r$ whence $a_j \smile r \leqq a_{j+1} \smile r \leqq a_j \smile r$, i.e.

$$a_j \smile r = a_{j+1} \smile r.$$

Summing up the two cases we obtain

$$a_j \smile r \leqq a_{j+1} \smile r \quad (j = 0, 1, \ldots, n-1).$$

This means that the chain in (7) is a maximal one and its length is at most n. It follows ([3], p. 104) that *the length of the interval $[r, v]$ for any relative complement r of a in $[u, v]$ cannot be greater than n.* The length of $R_u^v(a)$ is, a fortiori, at most n. Thus the theorem is proved.

Finally we prove

Theorem 3. *Every element of a partition lattice has both minimal and maximal complements.*

Proof. The customary proof of complementarity of partition lattices (see, e.g., [3], p. 148) proceeds, as is well-known, by giving effectively a complement to each element. It is easily seen that this complement is a minimal one. Thus existence of maximal complements is only to be shown.

Let A be an arbitrary element of a partition lattice P. In order to give a maximal complement M of A we begin by selecting one element from each A-class; the set of all selected elements will form a M-class. From the remaining elements of each A-class we select an element again; they will form another M-class. Then we continue constructing the M-classes in the same way (after we have selected all elements of an A-class, it will be left out of consideration at the construction of further M-classes). Clearly, M is a complement of A. Let C be any element of P greater than M. Then at least one of the C-classes is the set union of two or more M-classes and, therefore, it contains at least two elements belonging to the same A-class. Hence $A \frown C$ cannot be equal to the least element, formed by one-element classes, of P.

References

[1] G. Birkhoff, On the combination of subalgebras, *Proc. Cambridge Phil. Soc.*, **30** (1933), 441—464.
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