On Subobjects, Quotients, Kernels, Cokernels in a Partially Ordered Category

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1. Introduction. Preliminaries. Following MAC LANE [1] we call a category $\mathscr{B}$ a partially ordered category, if each set Rel (A, B) of morphisms $f\colon A \to B$ satisfies the following requirements:

(I-a) To each $f\colon A \to B$ there is a unique $f^{*}\colon B \to A$ with

$$f^{**} = f = ff^{*}f, \qquad (fg)^{*} = g^{*}f^{*}.$$

(I-b) Each Rel (A, B) is a modular lattice under a partial order relation "$\subset$" such that for $g, f\colon A \to B$, $g \subset f$ implies $g^{*} \subset f^{*}$, $gh \subset fh$.

(II-a) $hh^{*} \subset f^{*}f \cup 1$ implies $(f \cap g)h \supset fh \cap gh$.

(II-b) $hh^{*} \supset f^{*}f \cap 1$ implies $(f \cup g)h \subset fh \cup gh$.

(II-c) $f, g \in$ Rel (A, B) implies $f^{*}g \cap 1_A \subset f^{*}f \subset f^{*}g \cup 1_A$.

(II-d) $g \subset f$, $g^{*}g \cap 1 = f^{*}f \cap 1$, $gg^{*} \cup 1 = ff^{*} \cup 1$ implies $g = f$.

(III-a) For each pair of objects A, B there exist $N(A, B)$, $P(A, B)$ in Rel (A, B) such that $f \in$ Rel (A, B) implies $N \subset f \subset P$.

(III-b) $N(C, B)$, $P(A, C) = N(D, B)P(A, D)$, $NN = N$, $PP = P$.

(III-c) $NPN = N$, $PNP = P$.

These axioms are self-dual in the lattice sense.

They are valid in the standard model $\mathscr{M}$, which is the category of all (left) modules $A, B, \dots$ over a fixed ring and of morphisms all submodules of the direct sum $A \oplus B$ (these morphisms are called "aditive relations" in [1] and [2], "Korrespondenzen" in [5], "Correspondences" in [9], "homomorphic relations" in [8], "linear relations" in [4]).

In [5] D. PUPPE has defined a relation from the object A to the object B of an abelian category $\mathscr{A}$ as a subobject of $A \oplus B$ and has given (§ 2 in [5]) a construction of the category $\mathscr{K}(\mathscr{A})$ of relations over $\mathscr{A}$; $\mathscr{K}(\mathscr{A})$ satisfies all axioms of S. Mac Lane (§ 6.10 and § 9 in [5]).

We mention that, from a different point of view, P. HILTON has described the construction of the category of relations based on an abelian category by means of a fractional calculus ([9] and [10]).

A partially ordered category $\mathscr{B}$ is a particular "category with involution"; categories with involution have been firstly defined by Puppe ([5], § 1.3) and have been called "categories of relations" by H.-B. BRINKMANN ([6] and [7]). Indeed, if in the first group of axioms given by Mac Lane we cancel the equality $f=ff^{*}f$ and we weaken the assertion that each Rel (A, B) is a modular lattice by assuming merely that Rel (A, B) is partially ordered, then we obtain just the definition of a category with involution.

The axioms I, II, III from the definition of a partially ordered category $\mathscr{B}$ suffice to prove a number of basic properties valid for homomorphic relations or relations in an abelian category, but do not characterize the relations in an abelian category: in [5] "Beispiel A" from § 7 is a category with involution $\mathscr{K}$ which satisfies axioms I, II, III, but one could never endow the subcategory of "maps" ("eigentliche Morphismen" corresponding to usual maps or homomorphisms, defined by the conditions $f^{*}f\supset 1$, $ff^{*}\subset 1$) with a structure of pre-additive category!).

Finding all this, D. PUPPE has imposed ([5]) a set of other axioms on a category with involution which are sufficient conditions that the subcategory of "maps" be abelian and are necessarily satisfied by relations in an abelian category $\mathscr{A}$. The axioms due to D. Puppe, imply all axioms from [1] and are clearly more restrictive than the latter (we must except the condition of modularity for the lattice Rel (A, B) from I-b).

Here, we take an other point of view. Without requiring any other axiom, we remain within the frame of conditions I, II, III imposed on $\mathscr{B}$ by Mac Lane and we show that $\mathscr{B}$ can be embedded in a category with involution $\overline{\mathscr{B}}$ in which every "map" of $\mathscr{B}$ has a kernel and a cokernel. Our immediate aim is to show that the seemingly peculiar definitions of subobjects, quotients, kernels in $\mathscr{B}$ (due to Mac Lane) *amounts in* $\overline{\mathscr{B}}$ *to the usual definitions* with monomorphisms, epimorphisms and, respectively, projective limits; so, we are not concerned only with the question of the existence of kernels and cokernels for maps of $\mathscr{B}$, we give in Theorems 2 and 3 a *general characterization* of the "kernel" and the "cokernel" ("kernel" in the sense of [1]) of *an arbitrary morphism* of $\mathscr{B}$ (not necessarily of a map).

It is known that in $\mathscr{B}$ the symmetric idempotents $u\in \mathrm{Rel}\,(A, A)$, $u=u^{*}=uu$, form a sublattice which we denote by $\overline{\mathrm{Rel}}\,(A, A)$; all $s\subset 1_A$ and $q\supset 1_A$ are symmetric idempotents, called subobjects and quotients of A, respectively.

We recall also that for each object A there is a lattice isomorphism between the lattice of subobjects $s\subset 1_A$ and that of quotients $q\supset 1_A$, given by $\varphi(s)=1\cup sO^{*}$, $\psi(q)=1\cap qO$, $\varphi^{-1}=\psi$; $O(A, B)$ is defined by $O(A, B)=N(B, B)P(A, B)= =N(A, B)P(A, A)$. S. Mac Lane introduces for $f\colon A\to B$, $\mathrm{Def} f=f^{*}f\cap 1_A$, $\mathrm{Ker} f= =\psi(f^{*}f\cup 1)$ and shows that $\mathrm{Ker} f=1\cap f^{*}Nf$. We set $f^{*}f\cup 1=\mathrm{Coim}\, f$ and $\mathrm{Im} f= =\mathrm{Def} f^{*}=ff^{*}\cap 1_B$ and introduce also $\mathrm{Coker} f=\varphi(\mathrm{Im} f)$ so that $\mathrm{Coker} f=1_B\cup fPf^{*}$ (see also [3]).

2. Definition. We define the category $\overline{\mathscr{B}}$ associated to a partially ordered category $\mathscr{B}$ by

a) $\mathrm{Ob}\,\mathscr{B}\subset \mathrm{Ob}\,\overline{\mathscr{B}}$; for each $u\in\overline{\mathrm{Rel}}\,(A, A)$, $u=u_A\in \mathrm{Ob}\,\overline{\mathscr{B}}$; the new objects are but the symmetric idempotent morphisms of $\mathscr{B}$.

b) For $A, B\in \mathrm{Ob}\,\mathscr{B}$, $\mathrm{Hom}_{\overline{\mathscr{B}}}(A, B)=\mathrm{Rel}\,(A, B)$; for $B\in \mathrm{Ob}\,\mathscr{B}$ and $u\in\overline{\mathrm{Rel}}\,(A, A)$, $\mathrm{Hom}_{\overline{\mathscr{B}}}(u_A, B)=\mathrm{Hom}\,(u_A, B)=\{f\colon A\to B/fu=f\}$; for $A\in \mathrm{Ob}\,\mathscr{B}$ and $v\in\overline{\mathrm{Rel}}\,(B, B)$, $\mathrm{Hom}_{\overline{\mathscr{B}}}(A, v_B)=\mathrm{Hom}\,(A, v_B)=\{g\colon A\to B/vg=g\}$; $\mathrm{Hom}_{\overline{\mathscr{B}}}(u_A, v_B)=\mathrm{Hom}\,(u_A, v_B)= =\{f\colon A\to B/fu=vf=f\}$.

c) The composition of morphisms in $\bar{\mathscr{B}}$ is the same as in $\mathscr{B}$; we are allowed to state this because, if $f \in \mathrm{Rel}\,(A, B)$ and $g \in \mathrm{Hom}\,(B, u_C)$, then $gf \in \mathrm{Hom}\,(A, u_C)$ $(ug=g \Rightarrow ugf=gf)$, if $f \in \mathrm{Hom}\,(u_A, B)$ and $g \in \mathrm{Rel}\,(B, C)$, then $gf \in \mathrm{Hom}\,(u_A, C)$ $(fu=f \Rightarrow gfu=gf)$, if $f \in \mathrm{Hom}\,(u_A, B)$ and $g \in \mathrm{Hom}\,(B, u_C)$, then $gf \in \mathrm{Hom}\,(u_A, u_C)$, if $f \in \mathrm{Hom}\,(A, u_B)$ and $g \in \mathrm{Hom}\,(u_B, C)$, then $gf \in \mathrm{Rel}\,(A, C)$.

Remarks. 1. Denote by $\mathrm{Codef}\, f = \mathrm{Coim}\, f^* = ff^* \cup 1_B$; $f \in \mathrm{Hom}\,(u_A, B)$ if and only if $\mathrm{Def}\, f \subset u \subset \mathrm{Coim}\, f$; $f \in \mathrm{Hom}\,(A, v_B)$ if and only if $\mathrm{Im}\, f \subset v \subset \mathrm{Codef}\, f$.

Indeed, according to MAC LANE [1] $fu=f$ if and only if $f^*f \cap 1_A \subset u \subset f^*f \cup 1_A$.

2. The identity morphism in $\mathrm{Hom}\,(u_A, u_A)$ is just $u=u_A$ from $\overline{\mathrm{Rel}}\,(A, A)$.

3. One can easily check the validity of axioms I for $\bar{\mathscr{B}}$. For example, $f \in \mathrm{Hom}\,(u_A, B)$ implies $fu=f$ i.e. $uf^*=f^*$ and thus $f^* \in \mathrm{Hom}\,(B, u_A)$. If $f, g \in \mathrm{Hom}\,(u_A, B)$, then $f \cap g \in \mathrm{Hom}\,(u_A, B)$: the necessary and sufficient condition (given in [1]) for the distributive law $(f \cap g)h = fh \cap gh$, namely $hh^* \subset f^*f \cup 1_B$, can be applied in $(f \cap g)u$, as $uu^* = u \subset \mathrm{Coim}\, f$; then we have $(f \cap g)u = fu \cap gu = f \cap g$. Similarly, $(f \cup g)u = fu \cup gu = f \cup g$, as $uu^* \supset f^*f \cap 1_A$.

4. If we consider for each couple $A, B \in \mathrm{Ob}\,\mathscr{B}$ only those morphisms $f\colon A \to B$ for which we have $f^*f \supset 1_A$ and $ff^* \subset 1_B$, we obtain two subcategories $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively. The morphisms of $\mathscr{B}$ which satisfy simultaneously both inequalities form the subcategory $M(\mathscr{B})$ of "maps" in $\mathscr{B}$.

5. Corresponding to the above subcategories of $\mathscr{B}$, we have the subcategories $\bar{\mathscr{B}}_1$, and $\bar{\mathscr{B}}_2$ and $M(\bar{\mathscr{B}})$ of $\bar{\mathscr{B}}$:

$$\mathrm{Hom}_{\bar{\mathscr{B}}_1}(A, B) = \mathrm{Hom}_{\mathscr{B}_1}(A, B), \ \mathrm{Hom}_{\bar{\mathscr{B}}_2}(A, B) = \mathrm{Hom}_{\mathscr{B}_2}(A, B),$$
$$\mathrm{Hom}_{\bar{\mathscr{B}}_1}(u_A, B) = \{f\colon u_A \to B,\ f^*f \supset 1_{u_A} = u\},$$
$$\mathrm{Hom}_{\bar{\mathscr{B}}_1}(A, v_B) = \{g\colon A \to v_B,\ g^*g \supset 1_A\},$$
$$\mathrm{Hom}_{\bar{\mathscr{B}}_1}(u_A, v_B) = \{f\colon u_A \to v_B,\ f^*f \supset u\},$$
$$\mathrm{Hom}_{\bar{\mathscr{B}}_2}(u_A, B) = \{f\colon u_A \to B,\ ff^* \subset 1_B\},$$
$$\mathrm{Hom}_{\bar{\mathscr{B}}_2}(A, v_B) = \{g\colon A \to v_B,\ gg^* \subset v\},$$
$$\mathrm{Hom}_{\bar{\mathscr{B}}_2}(u_A, v_B) = \{g\colon u_A \to v_B,\ gg^* \subset v\}.$$

Clearly $\mathscr{B}$ is a full subcategory of $\bar{\mathscr{B}}$ and $M(\mathscr{B})$ is a full subcategory of $M\,(\bar{\mathscr{B}})$.

3. Theorems.

Theorem 1. *If $s \subset 1_A$ is a subobject of $A \in \mathrm{Ob}\,\mathscr{B}$, then the morphism $s \in \mathrm{Hom}\,(s_A, A)$ is a monomorphism in $\bar{\mathscr{B}}$. If $q \supset 1_A$ is a quotient of $A \in \mathrm{Ob}\,\mathscr{B}$, then the morphism $q \in \mathrm{Hom}\,(A, q_A)$ is an epimorphism in $\bar{\mathscr{B}}$.*

PROOF. It is clear that $s \in \mathrm{Hom}\,(s_A, A)$. If $sg=sh$ for $g, h \in \mathrm{Hom}\,(X, s_A)$, we must have also $sg=g$ and $sh=h$, so it follows obviously $g=h$.

Similarly $q \in \mathrm{Hom}\,(A, q_A)$ and from $gq=hq$, for $g, h \in \mathrm{Hom}\,(q_A, X)$ it follows $g=h=gq=hq$.

For $f\colon A \to B$ let $\mathrm{Ker}\, f = 1_A \cap f^* N(B, B) f$ be an object of the category $\bar{\mathscr{B}}$ and $i = 1_A \cap f^* N(B, B) f$ a morphism (from $\mathrm{Ker}\, f$ to A) in $\bar{\mathscr{B}}$.

Theorem 2. Ker f *and* i *have the following properties:*

(a) *For* $e=(ff^{*}\cup 1_B)O(A, B)$ *we have* $fi=ei$.

(a′) $\mathrm{Im}\, i\subset \mathrm{Ker} f$.

(b) *For each* $X\in \mathrm{Ob}\,\bar{\mathscr{B}}$ *and* $g\in \mathrm{Hom}\,(X, A)$ *with* $\mathrm{Im}\, g\subset \mathrm{Ker} f$ *(that is* $gg^{*}\cap 1_A\subset \subset 1_A\cap f^{*}Nf$*), there exists a unique morphism* $h\in \mathrm{Hom}\,(X, \mathrm{Ker} f)$, *such that* $ih=g$.

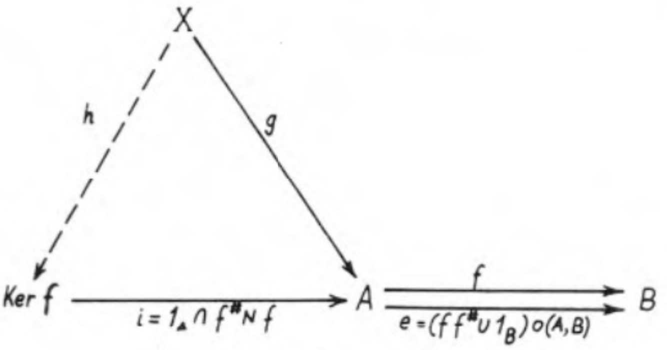

Fig. 1

(c) *If* $Y\in \mathrm{Ob}\,\bar{\mathscr{B}}$ *and* $j\in \mathrm{Hom}\,(Y, A)$ *have the properties:*

[$\bar{\text{a}}'$] $\mathrm{Im}\, j\subset \mathrm{Ker} f$;

[$\bar{\text{b}}$] $\forall X\in \mathrm{Ob}\,\bar{\mathscr{B}}$, $\forall g\in \mathrm{Hom}\,(X, A)$ *with* $\mathrm{Im}\, g\subset \mathrm{Ker} f$, $\exists$ *a unique* $h\in \mathrm{Hom}\,(X, Y)$ *such that* $jh=g$;

then there exists a unique isomorphism Θ *such that the following diagram is commutative*

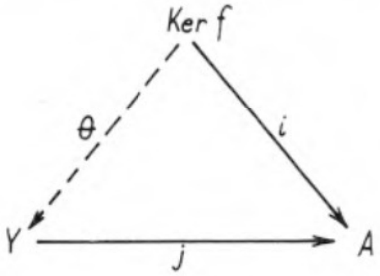

Fig. 2

(d) $\mathrm{Im}\, g\subset \mathrm{Ker} f\Rightarrow fg=eg$. *In the subcategory* $\bar{\mathscr{B}}_2$, $e=O(A, B)$ *and the opposite implication also holds, i.e.* $fg=eg\Leftrightarrow \mathrm{Im}\, g\subset \mathrm{Ker} f$ *which shows (see* (b)*) that in* $\bar{\mathscr{B}}_2$ $(\mathrm{Ker} f, i)$ *is a projective limit of the pair* $(f, O(A, A))$.

Proof. (a) Let us show firstly that $O(A, B)(1_A\cap f^{*}Nf)\subset f$. For $\forall f\colon A\to B$, $C\in \mathrm{Ob}\,\mathscr{B}$, the following equalities can be checked by calculation:

$$\mathrm{Ker}\,(N(B, C)f) = \mathrm{Def}\,(N(B, C)f) = \mathrm{Ker} f;$$

$$\mathrm{Ker}\,(P(B, C)f) = \mathrm{Def}\,(P(B, C)f) = \mathrm{Def} f.$$

Hence we may write

$$\mathrm{Def}\,[N(B, A)f(1_A \cap f^* Nf)] = \mathrm{Ker}\,[f(1_A \cap f^* Nf)];$$

besides,

$$\mathrm{Ker}\,[f(1_A \cap f^* Nf)] = (1_A \cap f^* Nf)f^* Nf(1_A \cap f^* Nf) \cap 1_A \supset$$

$$\supset (1_A \cap f^* Nf)(1_A \cap f^* Nf)(1_A \cap f^* Nf) \cap 1_A = 1_A \cap f^* Nf = \mathrm{Def}\,(1_A \cap f^* Nf).$$

If $f \in \mathrm{Rel}\,(A, B)$ and $g \in \mathrm{Rel}\,(A, B')$, then

$$\mathrm{Def}\, f \subset \mathrm{Def}\, g \Rightarrow O(B, C)f \subset O(B', C)g, \quad \forall C \in \mathrm{Ob}\,\mathscr{B}.$$

Indeed, $f^* f \cap 1_A \subset g^* g \cap 1_A$ implies $(f^* f \cap 1_A) \cup O^*(A, A) \subset (g^* g \cap 1_A) \cup O^*(A, A)$; constructing the dual of the equality $N(B, A)g = (g^* g \cup 1_A) \cap O(A, A)$, for $\forall g \in \in \mathrm{Rel}\,(A, B)$, given in [1], we obtain $(f^* f \cap 1_A) \cup O^*(A, A) = P(B, A)f$; so we have $P(B, A)f \subset P(B', A)g$; it follows $N(A, C)P(B, A)f \subset N(A, C)P(B', A)g \Rightarrow O(B, C)f \subset \subset O(B', C)g$.

Thus $\mathrm{Def}\,(1_A \cap f^* Nf) \subset \mathrm{Def}\,[N(B, A)f(1_A \cap f^* Nf)]$ implies

$$O(A, B)(1_A \cap f^* Nf) \subset O(A, B)N(B, A)f(1_A \cap f^* Nf) =$$
$$= N(B, B)f(1_A \cap f^* Nf) \subset 1_B f(f^* f \cap 1_A) = f.$$

(The inclusion $\mathrm{Ker}\, f \subset \mathrm{Def}\, f$ has been proved in [1].)

A theorem due to S. Mac Lane [1] says that $g \subset h \Leftrightarrow (\mathrm{Codef}\, h)g = h(\mathrm{Def}\, g)$; then $O(A, B)(1_A \cap f^* Nf) \subset f$ is equivalent to

$$(ff^* \cup 1_B)O(A, B)(1_A \cap f^* Nf) = f\,\mathrm{Def}\,[O(A, B)(1_A \cap f^* Nf)];$$

as $\mathrm{Def}\,[O(A, B)(1_A \cap f^* Nf)] = \mathrm{Ker}\,[P(A, B)(1_A \cap f^* Nf)] = \mathrm{Def}\,(1_A \cap f^* Nf) = 1_A \cap \cap f^* Nf$ we have just $ei = fi$.

(a′) $\mathrm{Im}\,(1_A \cap f^* Nf) = 1_A \cap f^* Nf = \mathrm{Ker}\, f$.

(b) Since $\mathrm{Im}\, g = \mathrm{Def}\, g^* \subset \mathrm{Ker}\, f = i \subset \mathrm{Coim}\, g^*$, we have $g^* i = g^* \Leftrightarrow ig = g$, so that $g \in \mathrm{Hom}\,(X, \mathrm{Ker}\, f)$; we can then consider $h = g$. Suppose that $h' \in \mathrm{Hom}\,(X, \mathrm{Ker}\, f)$, that is $ih' = h'$, and that h' satisfies also the condition of commutativity $ih' = g$. Then we obtain necessarily $h' = g$.

(c) $\mathrm{Im}\, j \subset \mathrm{Ker}\, f$ implies $\mathrm{Def}\, j^* \subset i \subset \mathrm{Coim}\, j^*$, so that $j^* i = j^* \Leftrightarrow ij = j$; it fo-lows that $j \in \mathrm{Hom}\,(Y, \mathrm{Ker}\, f)$. From $\mathrm{Im}\, j \subset \mathrm{Ker}\, f$ we obtain also, using (b), that there exists a unique morphism $h_1 \in \mathrm{Hom}\,(Y, \mathrm{Ker}\, f)$ such that $ih_1 = j$, namely just $h_1 = j$.

Applying [$\bar{\text{b}}$] to $X = \mathrm{Ker}\, f$ and $g = i$ ($\mathrm{Im}\, i = \mathrm{Ker}\, f$), we have that there exists a unique $h_2 \in \mathrm{Hom}\,(\mathrm{Ker}\, f, Y)$ with $jh_2 = i$. Let us show that this h_2 is an isomorphism.

The two conclusions obtained above for $h_1 = j$ and h_2, imply $h_1 h_2 \in \in \mathrm{Hom}\,(\mathrm{Ker}\, f, \mathrm{Ker}\, f)$, $h_2 h_1 \in \mathrm{Hom}\,(Y, Y)$ and

$$i(h_1 h_2) = (ih_1)h_2 = jh_2 = i, \quad j(h_2 h_1) = (jh_2)h_1 = ih_1 = j.$$

According to (b) for $X = \mathrm{Ker}\, f$ and $g = i$, there exists a unique $h_3 \in \in \mathrm{Hom}\,(\mathrm{Ker}\, f, \mathrm{Ker}\, f)$ which satisfies $ih_3 = i$, take namely $h_3 = i$; by Remark 2 $i = 1_{\mathrm{Ker}\, f}$, so that $h_1 h_2$ coincides with $1_{\mathrm{Ker}\, f}$. In the same way, if we apply [$\bar{\text{b}}$] to $X = Y$ and $g = j$ ($\mathrm{Im}\, j \subset \mathrm{Ker}\, f$), we have that there exists a unique $h_4 \in \mathrm{Hom}\,(Y, Y)$ which satisfies $jh_4 = j$, precisely $h_4 = 1_Y \in \mathrm{Hom}\,(Y, Y)$; it follows then $h_2 h_1 = 1_Y$.

We can now conclude that $h_2 \in \mathrm{Hom}\,(\mathrm{Ker}\, f,\ Y)$ is an isomorphism with $h_2^{-1}=$ $=h_1=j$.

(d) $\mathrm{Im}\, g \subset \mathrm{Ker}\, f \Leftrightarrow gg^{*} \cap 1_A \subset 1_A \cap f^{*} Nf \Rightarrow O(A, B)(gg^{*} \cap 1_A) \subset$ $\subset O(A, B)(1_A \cap f^{*} Nf) \subset f$.

$m=O(A, B)(gg^{*} \cap 1_A) \subset f$ implies $(\mathrm{Codef}\, f)m=f(\mathrm{Def}\, m)$, that is

$$(ff^{*} \cup 1_B)O(A, B)(gg^{*} \cap 1_A) = f(\mathrm{Def}\, m).$$

$$\mathrm{Def}\, m = \mathrm{Def}\,[O(A, B)(gg^{*} \cap 1_A)] = \mathrm{Ker}\,[P(A, B)(gg^{*} \cap 1_A)] = \mathrm{Def}\,(gg^{*} \cap 1_A) =$$

$$= gg^{*} \cap 1_A \quad \text{and} \quad (gg^{*} \cap 1_A)g = (\mathrm{Im}\, g)g = g.$$

Then it follows immediately

$$(ff^{*} \cup 1_B)O(A, B)(gg^{*} \cap 1_A)g = f(gg^{*} \cap 1_A)g \Leftrightarrow eg = fg.$$

If $ff^{*} \subset 1_B$, $e=1_B O(A, B)=O(A, B)$ and from $O(A, B)g=fg$ we have $O(A, B)gg^{*}=fgg^{*}$. Since $gg^{*} \subset 1$, $O(A, B)gg^{*} \subset O(A, B)$ and $fgg^{*} \subset f$, so that $fgg^{*} \subset O(A, B) \cap f=N(B, B)f$ (the last equality proved in [1]); thus $N(B, B)fgg^{*} \subset$ $\subset N(B, B)f$, which implies $\mathrm{Ker}\,(fgg^{*}) \subset \mathrm{Ker}\,(N(B, B)f)$. As $\mathrm{Ker}\,(fgg^{*})=$ $=\mathrm{Ker}\,[O(A, B)gg^{*}]=\mathrm{Def}\,(gg^{*})=gg^{*}$ and $\mathrm{Ker}\,(N(B, B)f)=\mathrm{Ker}\, f$, we have in fact $gg^{*} \subset \mathrm{Ker}\, f \Leftrightarrow \mathrm{Im}\, g \subset \mathrm{Ker}\, f$.

Thus in $\bar{\mathscr{B}}_2$, the inclusion $\mathrm{Im}\, g \subset \mathrm{Ker}\, f$ is quivalent to $fg=O(A, B)g$; then (a′) and (b) become: $fi=O(A, B)i$ and for each $g \in \mathrm{Hom}\,(X, A)$ with $fg=O(A, B)g$, there exists a unique $h \in \mathrm{Hom}\,(X, \mathrm{Ker}\, f)$ such that $ih= g$, which is just the necessary and sufficient condition for $(\mathrm{Ker}\, f, i)$ to be a projective limit of the pair $(f, O(A, B))$ (cf. [11], ch. 2.2 and ch. 3.1).

For $f\colon A \to B$ we consider $\mathrm{Coker}\, f=1_B \cup fP(A, A)f^{*}$ as an object of $\bar{\mathscr{B}}$ and $p=1_B \cup fP(A, A)f^{*}$ as a morphism from B to $\mathrm{Coker}\, f$ in $\bar{\mathscr{B}}$. Then, one can prove

Theorem 3. (a) *For* $t=O(A, B)(f^{*}f \cap 1_A)$ *we have* $pt=pf$.

(a′) $\mathrm{Coker}\, f \subset \mathrm{Coim}\, p$.

(b) $\forall X \in \mathrm{Ob}\, \bar{\mathscr{B}}$ *and* $g \in \mathrm{Hom}\,(B, X)$ *with* $\mathrm{Coker}\, f \subset \mathrm{Coim}\, g$ *(that is* $1_B \cup fPf^{*} \subset$ $\subset g^{*}g \cup 1_B$*), there exists a unique morphism* $h \in \mathrm{Hom}\,(\mathrm{Coker}\, f,\ X)$ *such that* $hp=g$

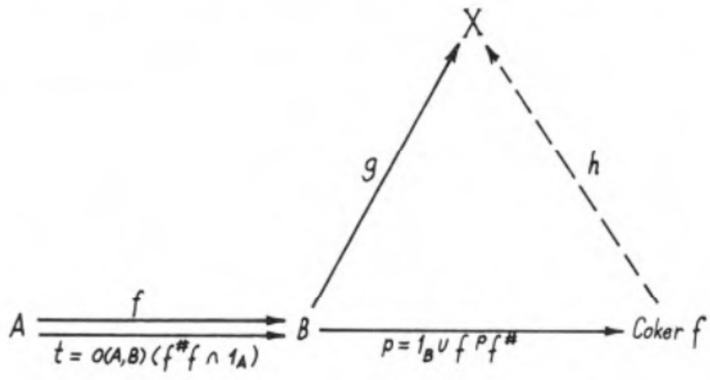

Fig. 3

(c) *If* $Y \in \text{Ob}\, \bar{\mathscr{B}}$ *and* $r \in \text{Hom}\,(B, Y)$ *have the properties:*

[$\bar{a}'$] $\text{Coker}\, f \subset \text{Coim}\, r$.

[$\bar{b}$] If $X \in \text{Ob}\, \bar{\mathscr{B}}$ *and* $g \in \text{Hom}\,(B, X)$ *with* $\text{Coker}\, f \subset \text{Coim}\, g$, $\exists$ *a unique* $h \in \text{Hom}\,(Y, X)$ *such that* $hr = g$;
then there exists a unique isomorphism $\Theta \in \text{Hom}\,(Y, \text{Coker}\, f)$ *such that* $\Theta r = p$

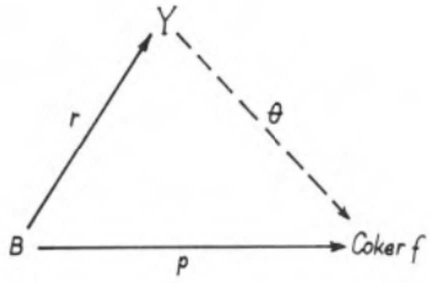

Fig. 4

(d) $\text{Coker}\, f \subset \text{Coim}\, g \Rightarrow gf = gt$. *In the subcategory* $\bar{\mathscr{B}}_1$, $t = O(A, B)$ *and* $gf = gt \Leftrightarrow \text{Coker}\, f \subset \text{Coim}\, g$, *so that* $(\text{Coker}\, f, p)$ *is the inductive limit of* $(f, O(A, B))$.

Corollary. For an arbitrary map $f: A \to B$ from $M(\mathscr{B})$, the pair $(f, O(A, B))$ has a kernel and a cokernel in $M(\bar{\mathscr{B}})$.

PROOF. $O(A, B)$ is a map in $\mathscr{B}$, $O(A, B) \in \text{Hom}_{M(\mathscr{B})}(A, B)$, as $O^*(A, B)O(A, B) = P(A, A) \supset 1_A$, $O(A, B)O^*(A, B) = N(B, B) \subset 1_B$. Needless to say that f and $O(A, B)$ are maps in $\bar{\mathscr{B}}$, too.

According to Theorem 2, $i = 1_A \cap f^* N f$: $(\text{Ker}\, f = i_A) \to A$ is a kernel of the pair $(f, O(A, B))$ in $\bar{\mathscr{B}}_2$; we are allowed to state the same fact in $M(\bar{\mathscr{B}})$, since not only $ii^* = i \subset 1_A$, but also $i^* i = i \supset 1_{\text{Ker}\, f} = i$ and thus $i \in \text{Hom}_{M(\bar{\mathscr{B}})}(\text{Ker}\, f, A)$ is a map in $\bar{\mathscr{B}}$.

According to Theorem 3, $p = 1_B \cap f N f^*: A \to (\text{Coker}\, f = p_B)$ is a cokernel of the pair $(f, O(A, B))$ in $\bar{\mathscr{B}}_1$; we are allowed to assert the same thing in $M(\bar{\mathscr{B}})$, since $p^* p = p \supset 1_B$, but also $pp^* = p \subset 1_{\text{Coker}\, f} = p$ and thus $p \in \text{Hom}_{M(\bar{\mathscr{B}})}(A, \text{Coker}\, f)$ is a map in $\bar{\mathscr{B}}$.

Theorem 4. *The category* $\bar{\mathscr{M}}$ *is equivalent to* $\mathscr{M}$.

PROOF. We define a functor $F: \mathscr{M} \to \bar{\mathscr{M}}$ by $F(A) = A$ for a module A and $F(R) = R$ for a linear relation R (submodule in a direct sum of modules $R \subseteq A \oplus B$). It is clear that the map induced by F from $\text{Rel}\,(A, B)$ to $\text{Hom}_{\bar{\mathscr{M}}}(A, B) = \text{Rel}\,(A, B)$ is the identity mapping. Moreover, each object $X \in \text{Ob}\, \bar{\mathscr{M}}$ is isomorphic to an object $F(A)$, $A \in \text{Ob}\, \mathscr{M}$; indeed, X is either a module $A = F(A)$, or a symmetric idempotent

$R_A \subseteq A \oplus A$; in the second situation, $it^*: \frac{\mathrm{Def}\, R}{\mathrm{Ker}\, R} \to R_A$ with $t: \mathrm{Def}\, R \to \frac{\mathrm{Def}\, R}{\mathrm{Ker}\, R}$ canonical projection, is an isomorphism in $\bar{\mathcal{M}}$: $R_A \subseteq A \oplus A$ symmetric idempotent submodule implies $R = \bigcup_{C \in \frac{\mathrm{Def}\, R}{\mathrm{Ker}\, R}} (C \times C)$, so that $R(it^*) = it^*$, $(ti^*)R = ti^*$, $(it^*)(ti^*) = R = 1_R$ and $(ti^*)(it^*) = 1_{\mathrm{Def}\, R/\mathrm{Ker}\, R}$.

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