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On α-close-to-convex functions

By HERB SILVERMAN (Charleston) and E. M. SILVIA (Davis)

Abstract. For $|\alpha| < \frac{\pi}{2}$, let $\mathcal{G}_\alpha$ denote the class of functions f, $f(0) = f'(0) - 1 = 0$, for which $\operatorname{Re} e^{i\alpha} f'(z) > 0$ in $\Delta = \{z : |z| < 1\}$. In this note, we discuss extremal, containment and convolution properties of $\mathcal{G}_\alpha$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in $\Delta = \{z : |z| < 1\}$. The subclasses of $\mathcal{A}$ consisting of functions that are univalent, starlike with respect to the origin, and convex will be denoted by $\mathcal{S}$, $\mathcal{S}t$, and $\mathcal{K}$, respectively. A function $f \in \mathcal{A}$ is close-to-convex, $f \in \mathcal{C}$, if there exists a function $g(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n$, $|c_1| = 1$, convex in Δ such that $\operatorname{Re} \frac{f'(z)}{g'(z)} > 0$, $z \in \Delta$. Goodman and Saff [3] classified subclasses of $\mathcal{C}$ in terms of the argument of c_1. To emphasize this distinction, we say a function $f \in \mathcal{A}$ is α *-close-to-convex*, denoted by $f \in \mathcal{C}_\alpha$, for α real with $|\alpha| < \frac{\pi}{2}$, if there exists a $g \in \mathcal{K}$ such that

$$\operatorname{Re} \frac{e^{i\alpha} f'(z)}{g'(z)} > 0, \; z \in \Delta. \tag{1.1}$$

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That $\mathcal{C}$ is a subset of $\mathcal{S}$ was shown by Kaplan [4]. In addition to illustrating that taking $\alpha = 0$ in (1.1) excludes functions that have the geometric properties shown by KAPLAN [4], GOODMAN and SAFF [3] obtained coefficient bounds for $\mathcal{C}_\alpha$ that were sharp only for $n = 2$.

Although our work was inspired by the paper by GOODMAN and SAFF [3], most of our discussion is concerned with the more manageable subclasses of α-close-to-convex functions that correspond to the convex function $g(z) = z$ in (1.1). For α real, $|\alpha| < \dfrac{\pi}{2}$, a function f is said to be in the class $\mathcal{G}_\alpha$ if

$$\operatorname{Re} e^{i\alpha} f'(z) > 0,\ \ z \in \Delta. \tag{1.2}$$

In the next two sections, we discuss extremal, containment, and convolution properties of the classes $\mathcal{G}_\alpha$. In addition, we show that any function f that is in $\mathcal{C}_\alpha$ for all α satisfying either $\frac{\pi}{2} - \epsilon < \alpha < \frac{\pi}{2}$ or $\frac{-\pi}{2} < \alpha < \frac{-\pi}{2} + \epsilon$ must be convex.

2. Extremal properties

Extremal information concerning $\mathcal{G}_\alpha$ follows from its relationship to the class $\mathcal{P}$ of functions in the form $p\,(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ that satisfy $\operatorname{Re} p\,(z) > 0$ for $z \in \Delta$. From (1.2), $f \in \mathcal{G}_\alpha$ if and only if there exists a $p \in \mathcal{P}$ such that

$$\frac{e^{i\alpha} f'(z) - i \sin\alpha}{\cos\alpha} = p\,(z). \tag{2.1}$$

This observation yields the extreme points for $\mathcal{G}_\alpha$.

Theorem 1. *The function f is in $\mathcal{G}_\alpha$ if and only if f can be expressed as*

$$f(z) = \int_X \left[-e^{-2i\alpha} z - (2e^{-i\alpha}\cos\alpha)\bar{x}\log(1 - xz)\right] d\mu(x) \tag{2.2}$$

where μ is a probability measure defined on the unit circle X. The extreme points of $\mathcal{G}_\alpha$ are the unit point masses

$$f_{x_0}(z) = -e^{-2i\alpha} z - (2e^{-i\alpha}\cos\alpha)\bar{x}_0 \log(1 - x_0 z),\ \ |x_0| = 1.$$

PROOF. From HERGLOTZ's Theorem [2], $p \in \mathcal{P}$ if and only if

$$p(z) = \int_X \frac{1+xz}{1-xz} d\mu(x)$$

for some probability measure μ on the unit circle X. In view of (2.1), $f \in \mathcal{G}_\alpha$ if and only if

$$f'(z) = e^{-i\alpha}\left[(\cos\alpha)\int_X \frac{1+xz}{1-xz} d\mu(x) + i\sin\alpha\right] \tag{2.3}$$

for μ a probability measure on X. We conclude that

$$f(z) = \int_0^z \left[\int_X \left(-e^{-2i\alpha} + \frac{2e^{-i\alpha}\cos\alpha}{1-x\zeta}\right) d\mu(x)\right] d\zeta$$

which yields (2.2) upon reversing the order of integration and integrating with respect to ζ.

Remark 1. From (2.3), we note that the derivatives of the extreme points for $\mathcal{G}_\alpha$ are the point masses $f_x'(z) = \dfrac{1+e^{2i\alpha}xz}{1-xz}$, for $|x| = 1$.

Several extremal properties are immediate.

Corollary 1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{G}_\alpha$, *then*

(i) $|a_n| \leq \dfrac{2}{n}\cos\alpha$, $n = 2, 3, 4, \cdots$,

(ii) $\dfrac{1-(2\cos\alpha)r+(\cos 2\alpha)r^2}{1-r^2} \leq \operatorname{Re} f'(z) \leq \dfrac{1+(2\cos\alpha)r+(\cos 2\alpha)r^2}{1-r^2}$, $|z| = r < 1$.

(iii) $\operatorname{Re} f'(z) > 0$ *for* $|z| < \dfrac{1}{\cos\alpha + |\sin\alpha|}$.

All bounds are sharp.

PROOF. The coefficient bounds are realized by the extreme points for $\mathcal{G}_\alpha$. We have

$$f_x(z) = z + \sum_{n=2}^{\infty} \frac{2e^{-i\alpha}\cos\alpha}{n} x^{n-1} z^n, \ |x| = 1,$$

from which (i) follows. To show (ii), it also suffices to consider the extreme points. The bilinear transformation

$$f_x'(z) = -e^{-i2\alpha} + (2\cos\alpha)\frac{e^{-i\alpha}}{1-xz}$$

maps the disk $|z| \le r$ onto the disk having center $-e^{-i2\alpha} + \dfrac{2(\cos\alpha)\, e^{-i\alpha}}{1-r^2}$ and radius $\dfrac{2r\cos\alpha}{1-r^2}$. Thus, for any x, $|x| = 1$,

$$(-\cos 2\alpha) + \frac{(2\cos\alpha)(\cos\alpha - r)}{1-r^2} \le \operatorname{Re} f_x'(z) \le (-\cos 2\alpha) + \frac{(2\cos\alpha)(\cos\alpha + r)}{1-r^2}$$

as needed.

Because $1-(2\cos\alpha)r+(\cos 2\alpha)r^2 > 0$ whenever $r < (\cos\alpha + |\sin\alpha|)^{-1}$, (iii) follows from (ii).

Corollary 2. *For each α, $0 < |\alpha| < \dfrac{\pi}{2}$, there exists an $f \in \mathcal{G}_\alpha$ such that $\min\limits_{|z|=r} \operatorname{Re} f' \to -\infty$ as $r \to 1^-$.*

This follows from the sharpness of the bound in the left side of the inequality given in (ii) because the numerator approaches $-2(\cos\alpha)(1-\cos\alpha)$ as $r \to 1^-$.

Remark 2. The bounds given in (ii) also follow from Robertson's discussion of extremal problems over $\mathcal{P}$ [9].

Next, we will show that no $\mathcal{G}_\alpha$ is contained in $\mathcal{G}_\beta$ for $\beta \ne \alpha$.

Theorem 2. *For each α, $|\alpha| < \dfrac{\pi}{2}$, there exists $f \in \mathcal{G}_\alpha$ such that $f \notin \mathcal{G}_\beta$ for $\beta \ne \alpha$.*

Proof. For the extreme point of $\mathcal{G}_\alpha$ given by

$$g(z) = -e^{-2i\alpha}z - (2e^{-i\alpha}\cos\alpha)\log(1-z), \tag{2.4}$$

we have

$$e^{i\beta}g'(z) = e^{i(\beta-\alpha)}\left((\cos\alpha)\left(\frac{1+z}{1-z}\right) + i\sin\alpha\right).$$

Setting $z = e^{i\theta}$, we get $e^{i\beta}g'(z) = ie^{i(\beta-\alpha)}\left[(\cos\alpha)\frac{\sin\theta}{1-\cos\theta} + \sin\alpha\right]$ and

$$\operatorname{Re} e^{i\beta}g'(e^{i\theta}) = -\sin(\beta-\alpha)\left(\frac{(\cos\alpha)(\sin\theta)}{1-\cos\theta} + \sin\alpha\right). \tag{2.5}$$

Since $\dfrac{\sin\theta}{1-\cos\theta} \to \infty$ when $\theta \to 0^+$ and $\dfrac{\sin\theta}{1-\cos\theta} \to -\infty$ as $\theta \to 0^-$, the right side of (2.5) tends to $-\infty$ as $\theta \to 0^+$ or $\theta \to 0^-$ when $\beta > \alpha$ or $\beta < \alpha$, respectively. Hence, $g \in \mathcal{G}_\alpha - \mathcal{G}_\beta$ for $\beta \neq \alpha$.

Theorem 2 tells us that the classes $\mathcal{G}_\alpha$ are different from each other. On the other hand, we know that the function $f(z) = z$ is in $\mathcal{G}_\alpha$ for all α, $|\alpha| < \dfrac{\pi}{2}$. Our next observation is a kind of converse to this result; expecting a function to be in all the $\mathcal{G}_\alpha$ for a "small" range of α will force the function to be $f(z) = z$.

Theorem 3. *Suppose that for some ϵ, $0 < \epsilon < \dfrac{\pi}{2}$, the function f is in $\mathcal{G}_\alpha$ for all α satisfying either*

$$\frac{\pi}{2} - \epsilon < \alpha < \frac{\pi}{2} \text{ or } \frac{-\pi}{2} < \alpha < \frac{-\pi}{2} + \epsilon.$$

Then $f(z) = z$.

Proof. Suppose $0 < \epsilon < \dfrac{\pi}{2}$ and $f \in \mathcal{G}_\alpha$ for $\dfrac{\pi}{2} - \epsilon < \alpha < \dfrac{\pi}{2}$. If $f'(z) = u(z) + iv(z)$, then $\operatorname{Re} e^{i\alpha}f'(z) = u(z)\cos\alpha - v(z)\sin\alpha$. For fixed $z \in \Delta$, let $\alpha \to \dfrac{\pi}{2}$. Then $\operatorname{Re} e^{i\alpha}f'(z) \to -v(z) \geq 0$ which implies that $v(z) \leq 0$. Since z was arbitrary, we conclude that $v(z) \leq 0$ for all $z \in \Delta$. Since $v(0) = 0$, the maximum principle yields that $v(z) \equiv 0$ throughout Δ. Thus, $f'(z) = u(z)$ and f' is constant. From $f(0) = 1$, we have $f(z) = z$. A similar argument, using the minimum principle, leads to the result when $\dfrac{-\pi}{2} < \alpha < \dfrac{-\pi}{2} + \epsilon$.

On the other hand, we can have many $f \in \mathcal{G}_\alpha$ for ranges of α bounded away from $\pm\dfrac{\pi}{2}$. Namely, we have the following

Theorem 4. *For each ϵ in the interval $\left(0, \dfrac{\pi}{2}\right)$, there exists a function f such that $f \in \mathcal{G}_\alpha$ for all $|\alpha| \leq \dfrac{\pi}{2} - \epsilon$ and $f \notin \mathcal{K}$.*

Proof. It is well known that $h(z) = z + \lambda z^n \in \mathcal{K}$ if and only if $|\lambda| \leq \dfrac{1}{n^2}$. For fixed ϵ, $0 < \epsilon < \dfrac{\pi}{2}$, choose an integer n such that $n > (\sin\epsilon)^{-1}$.

Since

$$\begin{aligned}\operatorname{Re} e^{i\alpha} f'(z) &= \operatorname{Re}\left\{e^{i\alpha}(1+(\sin\epsilon)z^{n-1}\right\} \geq \cos\alpha - |\sin\epsilon| \\ &\geq \cos\left(\frac{\pi}{2}-\epsilon\right) - \sin\epsilon = 0,\end{aligned}$$

$f \in \mathcal{G}_\alpha$. On the other hand, $\dfrac{\sin\epsilon}{n} > \dfrac{1}{n^2}$ implies that $f \notin \mathcal{K}$. Therefore, $f \in \mathcal{G}_\alpha - \mathcal{K}$ as needed.

Goodman and Saff [3] remarked that if $f \in \mathcal{C}_\alpha$ for all α, $|\alpha| < \dfrac{\pi}{2}$, then f is convex. Next, we will show that not the entire range of α is needed. The same small range of $|\alpha|$ near $\dfrac{\pi}{2}$ that restricted f to the identity function in Theorem 3 for $f \in \mathcal{G}_\alpha$ also requires $f \in \mathcal{C}_\alpha$ to consist only of convex functions. Note that the arbitrary function in $\mathcal{K}$ that defines $\mathcal{C}_\alpha$ is replaced by the identity function in the definition of $\mathcal{G}_\alpha$.

Theorem 5. *Suppose that for some ϵ, $0 < \epsilon < \dfrac{\pi}{2}$, the function f is in $\mathcal{C}_\alpha$ for all α satisfying either*

$$\frac{\pi}{2} - \epsilon < \alpha < \frac{\pi}{2} \text{ or } \frac{-\pi}{2} < \alpha < \frac{-\pi}{2} + \epsilon.$$

Then $f \in \mathcal{K}$.

Proof. Without loss of generality, for some fixed ϵ, $0 < \epsilon < \dfrac{\pi}{2}$, suppose that $f \in \mathcal{C}_\alpha$ for all α such that $\frac{\pi}{2} - \epsilon < \alpha < \frac{\pi}{2}$. Then there exists a sequence of reals $\{\beta_n\}_{n=1}^\infty$, $\dfrac{\pi}{2} - \epsilon < \beta_n < \dfrac{\pi}{2}$, converging to $\dfrac{\pi}{2}$, and a corresponding sequence of functions $\{g_n\}_{n=1}^\infty$ convex in Δ such that

$$\operatorname{Re} \frac{e^{i\beta_n} f'(z)}{g_n'(z)} > 0,\ z \in \Delta,\ n = 1, 2, 3, \dots .$$

Since $\mathcal{K}$ is a compact family [1], there exists a subsequence of $\{g_n\}_{n=1}^\infty$, $\{g_{n_k}\}_{k=1}^\infty$, that converges uniformly on compacta of Δ to some $g \in \mathcal{K}$. It follows that

$$\operatorname{Re} \frac{e^{i\beta_{n_k}} f'(z)}{g_{n_k}'(z)} \to \operatorname{Re} \frac{if'(z)}{g'(z)} \geq 0 \text{ as } k \to \infty.$$

Since $\operatorname{Re} \dfrac{if'(0)}{g'(0)} = 0$, an application of the minimum principle shows that $\dfrac{if'(z)}{g'(z)}$ is constant in Δ. Therefore, $f = g$.

Since the moduli of coefficients of functions in $\mathcal{G}_\alpha$ get small as α approaches $\frac{\pi}{2}$, it is natural to ask if the classes $\mathcal{G}_\alpha$ satisfy any other nice geometric properties that are enjoyed by other classes having similar coefficient restrictions. In the next section, we discuss how the classes $\mathcal{G}_\alpha$ relate to other subclasses and obtain some convolution results.

3. Inclusion properties

For fixed α, $|\alpha| < \frac{\pi}{2}$, and g given by (2.4), we have

$$1 + \frac{zg''(z)}{g'(z)} = \frac{1}{1-z} + \frac{e^{-2i\alpha}z}{1+e^{-2i\alpha}z}. \tag{3.1}$$

When $z_0 = -re^{2i\alpha}$,

$$\operatorname{Re}\left\{1 + \frac{z_0 g''(z_0)}{g'(z_0)}\right\} = \frac{1 + r\cos 2\alpha}{|1+re^{2i\alpha}|^2} - \frac{r}{1-r} \to -\infty$$

as $r \to 1^-$. Thus, $g \notin \mathcal{K}$. The determination of the radius of convexity for the classes $\mathcal{G}_\alpha$ is a very difficult problem. The solution for $\mathcal{G}_0$ was obtained by the first author [13]. The next theorem gives a bound on the radius of convexity for $f \in \mathcal{G}_\alpha$.

Theorem 6. *If R_{CV} denotes the radius of convexity for the class $\mathcal{G}_\alpha$, then*

$$\left(\sqrt{2}-1\right)(\cos\alpha + |\sin\alpha|)^{-1} \le R_{CV} \le \sqrt{1+\cos\alpha} - \sqrt{\cos\alpha}.$$

Proof. For the lower bound, the radius of convexity of $\mathcal{G}_0$ is known [7] to be $\sqrt{2}-1$. By (iii) of Corollary 1, we have that $f \in \mathcal{G}_\alpha$ implies that $\frac{f(\lambda z)}{\lambda} \in \mathcal{G}_0$ for $\lambda = (\cos\alpha + |\sin\alpha|)^{-1}$. It follows that $f \in \mathcal{G}_\alpha$ is convex at least for $|z| < (\sqrt{2}-1)(\cos\alpha + |\sin\alpha|)^{-1}$. From (3.1), we have that

$$1 + \frac{zg''(z)}{g'(z)} = 0 \text{ if and only if } z^2 - 2z - e^{2i\alpha} = 0$$

which gives the root $\sqrt{1+\cos\alpha} - \sqrt{\cos\alpha}$ as an upper bound on the radius of convexity.

Remark 3. For $\alpha = 0$ in the theorem, $R_{CV} = \left(\sqrt{2} - 1\right)$ which agrees with the result obtained earlier by the first author [13].

Krzyz [6] gave an example of a function that was in $\mathcal{G}_0$ and not starlike. It is natural to ask if $\mathcal{G}_\alpha \subset \mathcal{S}t$ for some α. To see that this is not the case, we again consider the extreme points of $\mathcal{G}_\alpha$, $\alpha \neq 0$. For g given in (2.4) and $0 < \theta < 2\pi$,

$$\operatorname{Re} \frac{g(e^{i\theta})}{e^{i\theta} g'(e^{i\theta})} = \frac{\sin \frac{\theta}{2}}{\cos\left(\frac{\theta - 2\alpha}{2}\right)} \left[\sin \alpha + (2 \cos \alpha) G(\theta)\right] \tag{3.2}$$

where $G(\theta) = (\sin \theta) \ln \left|2 \sin \frac{\theta}{2}\right| + \left(\dfrac{\pi - \theta}{2}\right) \cos \theta$. When $\theta = \pi + 2\alpha$ and $0 < |\alpha| < \dfrac{\pi}{2}$, we have that $\sin \dfrac{\theta}{2} = \cos \alpha > 0$ and

$$G(\theta) = (-\sin 2\alpha) \ln |2 \cos \alpha| + \alpha \cos 2\alpha.$$

As $\theta \to (\pi + 2\alpha)^+$, $\cos \frac{\theta - 2\alpha}{2} \to 0^-$ and, as $\theta \to (\pi + 2\alpha)^-$, $\cos \frac{\theta - 2\alpha}{2} \to 0^+$. Thus, for $\alpha \neq 0$, we may choose a value of $\theta > \pi + 2\alpha$ when $\sin \alpha + 2(\cos \alpha) G(\pi + 2\alpha) > 0$ and a value of $\theta < \pi + 2\alpha$ when $\sin \alpha + 2(\cos \alpha) G(\pi + 2\alpha) < 0$ to show that the right side of (3.2) can be negative. Thus, for any α, $0 < |\alpha| < \dfrac{\pi}{2}$, $g \notin \mathcal{S}t$.

Remark 4. The expression $\sin \alpha + (2 \cos \alpha) G(\pi + 2\alpha) = 0$ only when $\alpha = 0$. That the function g given by (2.4) is starlike when $\alpha = 0$ follows from Lemma 2 in a paper by Ruscheweyh [11]. As noted earlier, Krzyz [6] gave an example of a function that was in $\mathcal{G}_0 - \mathcal{S}t$; Mocanu [8] gave an example of a function $f \notin \mathcal{S}t$ satisfying the more restrictive condition $|f'(z) - 1| < 1$, $z \in \Delta$.

Our next theorem gives a condition under which polynomial members of $\mathcal{G}_\alpha$ are starlike and convex.

Theorem 7. *If $p_n(z) = z + \sum\limits_{k=2}^{n} c_k z^k \in \mathcal{G}_\alpha$, then $p_n \in \mathcal{S}t$ if $\cos \alpha \leq \dfrac{1}{2(n-1)}$ and $p_n \in \mathcal{K}$ if $\cos \alpha \leq \dfrac{1}{(n+2)(n-1)}$.*

Proof. It is known [5] that, for $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, the condition $\sum\limits_{k=2}^{\infty} k\, |a_k| \leq 1$ is sufficient for starlikeness of f, while $\sum\limits_{k=2}^{\infty} k^2\, |a_k| \leq 1$

is sufficient for the convexity of f. For $p_n(z) = z + \sum_{k=2}^{n} c_k z^k \in \mathcal{G}_\alpha$, the coefficient bounds from (i) of Corollary 1 yield

$$\sum_{k=2}^{n} k\,|c_k| \leq \sum_{k=2}^{n} 2\cos\alpha = 2(n-1)\cos\alpha \leq 1$$

when $\cos\alpha \leq \dfrac{1}{2(n-1)}$, and

$$\begin{aligned}\sum_{k=2}^{n} k^2\,|c_k| \leq \sum_{k=2}^{n} 2k\cos\alpha &= 2\left[\frac{n(n+1)}{2} - 1\right]\cos\alpha \\ &= (n+2)(n-1)\cos\alpha \leq 1\end{aligned}$$

whenever $\cos\alpha \leq \dfrac{1}{(n+2)(n-1)}$.

The previous theorem gives conditions on polynomial elements of $\mathcal{G}_\alpha$ that will yield the geometric properties of starlikeness and convexity. It would be nice to find other conditions or sets of conditions that would either do this for more general members of the $\mathcal{G}_\alpha$ or yield membership in other classes.

Next we turn to a discussion of some convolution properties of $\mathcal{G}_\alpha$. For $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $h(z) = \sum_{n=0}^{\infty} c_n z^n$ analytic in Δ, the convolution or Hadamard product of g and h is given by $(g * h)(z) = \sum_{n=0}^{\infty} b_n c_n z^n$. It follows from the next theorem that the classes $\mathcal{G}_\alpha$ are preserved under operators that can be realized as convolutions with specific convex functions.

Theorem 8. *If $f \in \mathcal{G}_\alpha$ and $\phi \in \mathcal{K}$, then $f * \phi \in \mathcal{G}_\alpha$.*

Proof. It is known [12] that if $\phi \in \mathcal{K}$, $g \in \mathcal{S}t$, and $F \in \mathcal{A}$, then $\dfrac{\phi * Fg}{\phi * g}(\Delta) \subset$ con $(F(\Delta))$ where "con" denotes the convex hull. For $f \in \mathcal{G}_\alpha$, take $g(z) = z$ and $F(z) = e^{i\alpha} f'(z)$. Then

$$\frac{\phi * Fg}{\phi * g}(z) = \frac{\phi(z) * e^{i\alpha} z f'(z)}{z} = e^{i\alpha}(\phi * f)'(z).$$

From $\operatorname{Re} e^{i\alpha} f'(z) > 0$, we have that $\operatorname{Re} e^{i\alpha}(\phi * f)'(z) > 0$. Therefore, $\phi * f \in \mathcal{G}_\alpha$.

Particular choices of convex functions give us some well known property preserving operators. Two examples are offered in the following

Corollary 3. *If $f \in \mathcal{G}_\alpha$, then so are*

$$F_1(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt, \ \operatorname{Re}\gamma > 0$$

and

$$F_2(z) = \int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta} d\zeta, \ |x| \le 1, \ x \ne 1.$$

Proof. Observe that $F_j(z) = (h_j * f)(z)$, $j = 1, 2$, for $h_1(z) = \sum_{n=1}^{\infty} \frac{1+\gamma}{n+\gamma} z^n$, $\operatorname{Re}\gamma > 0$, and $h_2(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{n(1-x)} z^n = \frac{1}{1-x} \log\left[\frac{1-xz}{1-z}\right]$. Since h_1 was shown to be convex by Ruscheweyh [10] and h_2 is clearly convex, the result follows from Theorem [8].

According to the next result, the class $\mathcal{G}_\alpha$ is closed under convolution for some, but not all, α.

Theorem 9. *There exists a real number $\alpha_0 \approx 1.24773$ for which $\mathcal{G}_\alpha$ is closed under convolution when $\alpha_0 \le |\alpha| < \frac{\pi}{2}$ and $\mathcal{G}_\alpha$ is not closed under convolution when $0 \le |\alpha| < \alpha_0$.*

Proof. First, we show that it suffices to consider the extreme point g that is given by (2.4). For $f, h \in \mathcal{G}_\alpha$, there exist probability measures μ_1 and μ_2 on the unit circle X such that

$$f(z) = \int_X \left(z + \sum_{n=2}^{\infty} \frac{2e^{-i\alpha}\cos\alpha}{n} x^{n-1} z^n \right) d\mu_1(x)$$

and

$$h(z) = \int_X \left(z + \sum_{n=2}^{\infty} \frac{2e^{-i\alpha}\cos\alpha}{n} y^{n-1} z^n \right) d\mu_2(y).$$

Thus,

$$\begin{aligned}(f * h)(z) &= \int_{X\times X} \left(z + \sum_{n=2}^{\infty} \frac{4e^{-2i\alpha}\cos^2\alpha}{n^2} x^{n-1} y^{n-1} z^n \right) d\mu_1(x) d\mu_2(y) \\ &= \int_X \left(z + \sum_{n=2}^{\infty} \frac{4e^{-2i\alpha}\cos^2\alpha}{n^2} (w)^{n-1} z^n \right) d\mu(w)\end{aligned}$$

for $|w| = 1$ and μ is a probability measure on X. We conclude that $\operatorname{Re}\left(e^{i\alpha}(f*h)'(z)\right) > 0$ whenever

$$\operatorname{Re}\left\{e^{i\alpha}\left(1+\sum_{n=2}^{\infty}\frac{4e^{-2i\alpha}\cos^2\alpha}{n}z^{n-1}\right)\right\} = \operatorname{Re} e^{i\alpha}(g*g)'(z) > 0.$$

For g given by (2.4),

$$e^{i\alpha}(g*g)'(z) = -e^{i\alpha} - 2e^{-2i\alpha}\cos\alpha - 4e^{-i\alpha}\left(\cos^2\alpha\right)\frac{\log(1-z)}{z}.$$

When $z = e^{i\theta}$, $0 < \theta < 2\pi$,

$$\operatorname{Re}\left\{e^{i\alpha}(g*g)'(z)\right\} = (\cos\alpha)\left[1 - 4\cos^2\alpha - 4(\cos\alpha)H(\alpha,\theta)\right],$$

where

$$H(\alpha,\theta) = \left((\cos(\alpha+\theta))\log(2\sin\frac{\theta}{2}) - \left(\frac{\pi-\theta}{2}\right)\sin(\alpha+\theta)\right).$$

Consequently, $\operatorname{Re}\left\{e^{i\alpha}(g*g)'(z)\right\}$ is positive if and only if

$$G(\alpha,\theta) = \left[1 - 4\cos^2\alpha - 4(\cos\alpha)H(\alpha,\theta)\right]$$

is positive. We can determine ranges of α for which $\mathcal{G}_\alpha$ is not closed under convolution by choosing specific θ for which $G(\alpha,\theta) < 0$. Since $G(-\alpha, 2\pi-\theta) = G(\alpha,\theta)$, we have that $\mathcal{G}_\alpha$ is closed under convolution if and only if $\mathcal{G}_{-\alpha}$ is closed under convolution. It is easy to verify that $G(\alpha,\pi) = 1 - 4(1-\ln 2)\cos^2\alpha \le 0$ for $|\alpha| \le \arccos\left(2\sqrt{1-\ln 2}\right) \approx .44$, $G(\alpha,\frac{\pi}{2}) = (-1+\frac{\pi}{2}) + (\ln 2)\sin 2\alpha + (-2+\frac{\pi}{2})\cos 2\alpha < 0$ for $\alpha \in (-.90, -.12)$, and $G(\alpha, \frac{\pi}{10}) < 0$ for α in $(-1.22, -.85)$. Additional choices of θ show that $G(\alpha,\theta)$ can be negative for $|\alpha| \le 1.2477$.

On the other hand, by fixing α and varying θ, we can verify numerically that $G(\alpha,\theta) \ge 0$ for $|\alpha| \ge \alpha_0 \approx 1.2477^+$ and all $\theta \in (0, 2\pi)$. This completes the proof.

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HERB SILVERMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHARLESTON
CHARLESTON, SC 29424
E-mail: silvermanh@cofc.edu

E. M. SILVIA
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA AT DAVIS
DAVIS, CA 95616–8633
E-mail: emsilvia@ucdmath.ucdavis.edu

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