On stable rings

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The stable torsion theories (i.e. the torsion class of which is closed under injective hulls) appeared in the literature in many situations and play an important rôle. It is therefore natural to ask what are the rings having the stable torsion theories only. The present paper is meant as an introduction to the study of such rings.

1. Preliminaries

All the rings considered below are assumed to be associative rings with identity and R-mod will denote the category of all unitary left R-modules. A ring R is said to be subcommutative if $Ra=aR$ for all $a\in R$. By a local ring we mean a ring having exactly one proper maximal left ideal. A non-zero element a is said to be a left zero divisor if $ab=0$ for some $b\neq 0$. A commutative ring without zero divisors will be called a domain. For any subset S of an R-module M we define the left annihilator of S by $(0\colon S)=\{\lambda \mid \lambda\in R,\ \lambda S=0\}$. For our further purposes it is convenient to consider 0 as a minimal left ideal and R as a maximal left ideal and a prime ideal. Recall that a ring R is hereditary if every left ideal is a projective left module. In this case the class of all injective modules is closed under homomorphic images (see [6]). A ring direct sum will be denoted by $\dotplus$, while that of modules by $\oplus$. The injective hull of a module M is denoted by $\hat{M}$. A module M is said to be injective with respect to an inclusion $A\subseteq B$ if every homomorphism $A\to M$ can be extended to B.

An idempotent radical r is a subfunctor of the identity functor for R-mod such that $r(r(M))=r(M)$ and $r(M/r(M))=0$ for all $M\in R$-mod. A torsion theory $(\mathfrak{M}, \mathfrak{B})$ for R-mod is an ordered pair of classes of modules orthogonal with respect to the functor Hom, i.e. $A\in\mathfrak{M}$ iff $\mathrm{Hom}_R(A, B)=0$ for all $B\in\mathfrak{B}$ and $B\in\mathfrak{B}$ iff $\mathrm{Hom}_R(A, B)=0$ for all $A\in\mathfrak{M}$. The class $\mathfrak{M}$ is called the torsion class and $\mathfrak{B}$ the torsion-free class. As it is well-known (see e.g. [7]) there is a one-to-one correspondence between torsion theories and idempotent radicals. A torsion theory $(\mathfrak{M}, \mathfrak{B})$ will be called hereditary if the class $\mathfrak{M}$ is closed under submodules. If $(\mathfrak{M}, \mathfrak{B})$ is a hereditary torsion theory then the system $\mathfrak{F}=\{I \mid I\subseteq R \text{ a left ideal, } R/I\in\mathfrak{M}\}$ is a radical filter, i.e. the following conditions are satisfied:

(F_1) If $I\in\mathfrak{F}$, $I\subseteq K$, a left ideal then $K\in\mathfrak{F}$.
(F_2) If $I\in\mathfrak{F}$ and $\lambda\in R$ then $(I\colon\lambda)\in\mathfrak{F}$.
(F_3) If $I\subseteq K$, $K\in\mathfrak{F}$ and $(I\colon\varkappa)\in\mathfrak{F}$ for all $\varkappa\in K$ then $I\in\mathfrak{F}$.

Conversely, if $\mathfrak{F}$ is a radical filter then $(\{M|(0: m)\in\mathfrak{F}$ for all $m\in M\}$, $\{N|(0: m)\notin\mathfrak{F}$ for all $m\in N$, $m\neq 0\}$ is a hereditary torsion theory and this correspondence is one-to-one (see e.g. [20] for details). If $\mathfrak{F}$ is a radical filter then $\mathfrak{F}$ is closed under finite intersections and products and by $\mathfrak{F}^*$ we shall denote $\{I|I$ a left ideal with $(I: \lambda)\notin\mathfrak{F}$ for all $\lambda\in R\dot{-}I\}$ (the set $\mathfrak{F}^*$ is called the cofilter corresponding to $\mathfrak{F}$). Every class $\mathfrak{C}$ of modules defines uniquely a torsion theory $(\mathfrak{M}, \mathfrak{B})$, namely $\mathfrak{B}=\{B|\operatorname{Hom}_R(C, B)=0$ for all $C\in\mathfrak{C}\}$ and $\mathfrak{M}=\{A|\operatorname{Hom}_R(A, B)=0$ for all $B\in\mathfrak{B}\}$. This torsion theory is said to be generated by the class $\mathfrak{C}$. The torsion theory cogenerated by $\mathfrak{C}$ is defined dually. A submodule $N\subseteq M$ of a module M is called essential if $N\cap A=0$ implies $A=0$ for each submodule $A\subseteq M$. The hereditary torsion theory corresponding to the least radical filter containing all the essential left ideals will be called the GOLDIE's torsion theory (see [10]).

2. Basic properties

In this section we introduce the notion of stable ring and give some basic properties. The following proposition has already apperad in the literature in various forms. However, we include it with proof for the sake of completeness and for the convenience of the reader.

2.1. Proposition. Let $(\mathfrak{M}, \mathfrak{B})$ be a torsion theory for R-mod and let r be the corresponding idempotent radical. Then the following are equivalent:

(i) If $M\in\mathfrak{M}$ then $\hat{M}\in\mathfrak{M}$.
(ii) If I is injective then $r(I)$ is so.
(iii) If $M\in\mathfrak{M}$ and $\hat{M}/M\in\mathfrak{B}$ then M is injective.
(iv) If $M\in\mathfrak{M}$ and $\operatorname{Ext}_R(N, M)=0$ for all $N\in\mathfrak{M}$ then M is injective.

Moreover, if $(\mathfrak{M}, \mathfrak{B})$ is hereditary and $\mathfrak{F}$ denotes the corresponding radical filter then these conditions are equivalent to the following (see [21]).

(v) If $I\subseteq K$ are left ideals and $K\in\mathfrak{F}^*$, $(I: \lambda)\in\mathfrak{F}$ for all $\lambda\in K$ then $I=K\cap L$ for some $L\in\mathfrak{F}$.

PROOF. (i) implies (ii). We have $r(I)\in\mathfrak{M}$ and hence $\widehat{r(I)}\in\mathfrak{M}$. But $r(I)\subseteq\widehat{r(I)}\subseteq I$ and so $r(I)=\widehat{r(I)}$.

(ii) implies (iii). Since $M\in\mathfrak{M}$ and $\hat{M}/M\in\mathfrak{B}$, $M=r(\hat{M})$.

(iii) implies (iv). Let $M\in\mathfrak{M}$ be such that $\operatorname{Ext}_R(N, M)=0$ for all $N\in\mathfrak{M}$. Denoting $r(\hat{M}/M)=A/M$ we get $\operatorname{Ext}_R(A/M, M)=0$ and hence $A=M$ since M is essential in A. Thus $\hat{M}/M\in\mathfrak{B}$ and we may use (iii).

(iv) implies (i). Let $M\in\mathfrak{M}$. The exact sequence

$$0\to r(\hat{M})\to\hat{M}\to\hat{M}/r(\hat{M})\to 0$$

induces for each $N\in\mathfrak{M}$ the exact sequence

$$0=\operatorname{Hom}_R(N, \hat{M}/r(\hat{M}))\to\operatorname{Ext}_R(N, r(\hat{M}))\to\operatorname{Ext}_R(N, \hat{M})=0.$$

Hence, by (iv), $r(\hat{M})$ is injective and so $\hat{M}=r(\hat{M})$ (as $M\subseteq r(\hat{M})$).

Now suppose that the torsion theory $(\mathfrak{M}, \mathfrak{B})$ is hereditary. (i) implies (v). Let L/I be maximal with respect to $L/I\cap K/I=0$. Then $L\cap K=I$ and $K/I\cong(K+L)/L$, so $(K+L)/L\in\mathfrak{M}$ (since $(I:\lambda)\in\mathfrak{F}$ for all $\lambda\in K$ implies $K/I=r(R/I)$). We claim that $(K+L)/L$ is essential in R/L. Indeed, if $S/L\cap(K+L)/L=0$ then $S\cap(K+L)=L$. However, $(L+S)\cap K=I$, and consequently $(L+S)/I\cap K/I=0$. From the maximality of L/I we see $(L+S)/I=L/I$, i.e. $S=L$. Now we have $(K+L)/L\subseteq R/L\subseteq\widehat{(K+L)/L}$ and $\widehat{(K+L)/L}\in\mathfrak{M}$, and consequently $R/L\in\mathfrak{M}$, i.e. $L\in\mathfrak{F}$.

(v) implies (i). Let $M\in\mathfrak{M}$ and $x\in\hat{M}\dot{-}r(\hat{M})$. Put $I=(0:x)$, $K=(r(\hat{M}):x)$. Since $r(R/K)=r(R(x+r(\hat{M})))=0$, $K\in\mathfrak{F}^*$. If $\varkappa\in K$ then $(I:\varkappa)=(0:\varkappa x)\in\mathfrak{F}$ since $\varkappa x\in r(\hat{M})$. Hence there is $L\in\mathfrak{F}$ such that $I=K\cap L$. We show that K/I is essential in R/I. Indeed, if $a\in R$, $a\notin K$ then $ax\notin r(\hat{M})$. On the other hand, $M\subseteq r(\hat{M})\subseteq\hat{M}$, so $r(\hat{M})$ is essential in $\hat{M}$ and there is $s\in R$ with $sax\neq 0$, $sax\in r(\hat{M})$. Therefore $sa\notin I$ and $sa\in K$, i.e. $s(a+I)\neq 0$ and $s(a+I)\in K/I$. Now we have $K/I\cap L/I=0$ and the essentiallity of K/I yields $L=I$, a contradiction. Thus $\hat{M}=r(\hat{M})$.

2.2. Definition. A torsion theory $(\mathfrak{M}, \mathfrak{B})$ satisfying the equivalent conditions of the preceding proposition will be called stable.

2.3. Remark. It follows immediately from Proposition 2.1 that every torsion theory containing the Goldie's torsion theory is stable.

2.4. Definition. A ring R is said to be stable (h-stable) if any (hereditary) torsion theory for R-mod is stable.

2.5. Proposition. Let $R=T_1\dot{+}T_2\dot{+}\dots\dot{+}T_n$ be a direct sum of rings. Then R is stable (h-stable) if and only f T_i are stable (h-stable) for all $i=1, 2, \dots, n$.

Proof. The proof is based on the structure of modules over a direct sum of rings as explained e.g. in § 9, Ch. 9 of [19] and has a technical character. The details are left to the reader.

2.6. Lemma. *Let R be a h-stable ring and r an idempotent radical for* R-mod. *If $I\in R$*-mod *is injective then $r(I)$ is injective iff $I/r(I)$ is so.*

Proof. Let $I/r(I)$ be injective. Denote by $\mathfrak{A}$ the smallest torsion-free class containing $I/r(I)$. It is an easy exercise to show that the corresponding torsion class is closed under submodules. Hence the radical s corresponding to $\mathfrak{A}$ is hereditary. Since $I/r(I)\in\mathfrak{A}$, $s(I/r(I))=0$, i.e. $s(I)\subseteq r(I)$. The converse is obvious and therefore $s(I)=r(I)$. But $s(I)$ is a direct summand of I by the hypothesis.

2.7. Proposition. Let R be a hereditary ring. Then R is stable if and only if R is h-stable.

Proof. It suffices to use 2.6 and 2.1.

3. Hereditary rings and stability

If $M \in R$-mod then $k(M)$ will denote the set of all $\lambda \in R$ such that M is injective with respect to the inclusion $R\lambda \subseteq R$.

3.1. Lemma. *Let* $M \in R$-mod *and* $\lambda \in R$. *Then the following are equivalent:*
(i) $\lambda \in k(M)$,
(ii) *If* $m \in M$ *and* $(0: \lambda)m = 0$ *then* $m \in \lambda M$.

Proof. Obvious.

3.2. Lemma. *Let* $M \in R$-mod. *Then*
(i) $0, 1 \in k(M)$.
(ii) *If* $\lambda, \varrho \in R$, $R\lambda = R\varrho$ *then* $\lambda \in k(M)$ *iff* $\varrho \in k(M)$.
(iii) *If* $\lambda_1, \lambda_2, \dots, \lambda_n \in k(M)$ *and* $R\lambda_i$ *are maximal left ideals then* $\lambda_1 \cdot \lambda_2 \cdot \cdots \lambda_n \in \in k(M)$.

Proof. The statements (i) and (ii) are obvious. For (iii) we use the induction by n. Let $m \in M$ and $(0: \lambda_1 \cdot \lambda_2 \cdots \lambda_n)m = 0$. Since $(0: \lambda_1) \subseteq (0: \lambda_1 \cdot \lambda_2 \cdots \lambda_n)$, $m = \lambda_1 a$, $a \in M$ suitable. Firstly suppose $(0: \lambda_2 \cdots \lambda_n) \subseteq R\lambda_1$. If $\varrho \in (0: \lambda_2 \cdots \lambda_n)$ then $\varrho = \sigma\lambda_1$ for some $\sigma \in R$, and hence $\sigma \in (0: \lambda_1 \cdot \lambda_2 \cdots \lambda_n)$. Therefore $(0: \lambda_2 \cdots \lambda_n) \subseteq (0: \lambda_1 \lambda_2 \cdots \lambda_n) \cdot \lambda_1$, and consequently $(0: \lambda_2 \cdots \lambda_n) \cdot a = 0$. By the induction hypothesis there is a $b \in M$ with $a = \lambda_2 \cdots \lambda_n \cdot b$, and so $m = \lambda_1 \lambda_2 \cdots \lambda_n b$. Let now $(0: \lambda_2 \cdots \lambda_n) \subsetneqq R\lambda_1$. Then we have $R\lambda_1 + (0: \lambda_2 \cdots \lambda_n) = R$ since $R\lambda_1$ is maximal. So $1 = \varrho\lambda_1 + \mu$, $\varrho \in R$ and $\mu \in (0: \lambda_2 \cdots \lambda_n)$. Hence $\lambda_2 \cdots \lambda_n = \varrho\lambda_1 \lambda_2 \cdots \lambda_n$ (by multiplying of the last equality by $\lambda_2 \cdots \lambda_n$ on the right). From this, $R\lambda_2 \cdots \lambda_n = R\lambda_1 \lambda_2 \cdots \lambda_n$ and $\lambda_1 \lambda_2 \cdots \lambda_n \in k(M)$ by (ii) and by the induction hypothesis.

3.3. Lemma. *Let* $(\mathfrak{M}, \mathfrak{B})$ *be a torsion theory and* $\lambda \in R$ *be such that* $R\lambda \subseteq \lambda R$ *is a maximal left ideal. Let* $M \in \mathfrak{M}$ *and* $\operatorname{Ext}_R(N, M) = 0$ *for every* $N \in \mathfrak{M}$. *Then* $\lambda \in k(M)$ *provided at least one of the following conditions for* M *holds:*
(i) *Each submodule of* M *belongs to* $\mathfrak{M}$.
(ii) *The projective homological dimension of every factor-module of* M *is at most* 1.
(iii) *The injective homological dimension of* M *is at most* 1.

Proof. Assume $\lambda \notin k(M)$. Then there is an $m \in M$ such that $(0: \lambda)m = 0$ and $m \notin \lambda M$. Since $R\lambda \subseteq \lambda R$, λM is a submodule of M. Put $N = R(m + \lambda M)$. We have $N \cong R/(\lambda M: m)$. However, $R\lambda \subseteq (\lambda M: m) \subsetneqq R$. Hence the maximality of $R\lambda$ yields $(\lambda M: m) = R\lambda$. If M satisfies (i), $Rm + \lambda M \in \mathfrak{M}$ and consequently

$$N \cong (Rm + \lambda M)/\lambda M \in \mathfrak{M}.$$

Then $\operatorname{Ext}_R(N, M) = 0$. If (ii) or (iii) holds then the same gives the exact sequence

$$0 = \operatorname{Ext}_R(M/\lambda M, M) \to \operatorname{Ext}_R(N, M) \to \operatorname{Ext}^2_R\big(M/(Rm + \lambda M), M\big) = 0.$$

Since $\operatorname{Ext}_R(N, M) = \operatorname{Ext}_R(R/R\lambda, M) = 0$, M is injective with respect to $R\lambda \subseteq R$, a contradiction.

3.4. Theorem. *Let R be a left hereditary left noetherian ring. Then R is stable, provided the following condition is satisfied:* (α) *If $\lambda \in R$, $\lambda \neq 0$, then there are $\lambda_1, \dots, \lambda_n \in R$ such that $R\lambda = R\lambda_1 \lambda_2 \cdots \lambda_n$, $R\lambda_i \subseteq \lambda_i R$ and $R\lambda_i$ is a maximal left ideal for every i.*

Proof. Let $(\mathfrak{M}, \mathfrak{B})$ be a torsion theory for R-mod and $M \in \mathfrak{M}$ be such that $\mathrm{Ext}_R(N, M) = 0$ for each $N \in \mathfrak{M}$. Since R is left hereditary, the condition (iii) from 3.3 is satisfied trivially. Now from (α), 3.2 and 3.3 we see that $k(M) = R$. Hence M is injective with respect to the inclusions of all the principal left ideals. But such a module over a left hereditary left noetherian ring is injective (see [11], Theorem 6.1). An application of 2.1 finishes the proof.

3.5. Theorem. *Let R be a left hereditary ring without zero divisors and let the condition* (α) *from the preceding theorem be satisfied. Then R is stable.*

Proof. Using Theorem 6.3 of [11] we can proceed in the similar way as in the proof of the preceding theorem.

4. Some sufficient conditions for stability

4.1. Lemma. *Let R be such a ring that $J(R)/(J(R))^2$ is a simple module, where $J(R)$ is the Jacobson radical of R. Then $J(R)$ is a principal left ideal and $(J(R))^n/(J(R))^{n+1}$ is a simple module for every $n = 1, 2, \dots$, provided at least one of the following conditions is satisfied:*

(i) *$J(R)$ is nilpotent.*

(ii) *$J(R)$ is a finitely generated left ideal.*

Proof. Firstly we show that the condition (i) (under our hypothesis) implies the condition (ii). Since $J(R)/(J(R))^2$ is simple, $J(R) = (J(R))^2 + Ra$ for each $a \in J(R) \dot{-} (J(R))^2$ (if $J(R) = (J(R))^2$ then the nilpotence of $J(R)$ yields $J(R) = 0 = R \cdot 0$). Then $(J(R))^2 = (J(R))^4 + (J(R))^2 \cdot Ra + Ra(J(R))^2 + RaRa \subseteq (J(R))^3 + RJ(R)a \subseteq (J(R))^3 + Ra^2 \subseteq (J(R))^2$. Similarly, $(J(R))^n = (J(R))^{n+1} + Ra^n$. However, $(J(R))^n = 0$ for some n and so $J(R) = Ra^n + Ra^{n-1} + \dots + Ra$. Now we have $J(R)(J(R)/Ra) = ((J(R))^2 + Ra)/Ra = J(R)/Ra$ and the Nakayma's Lemma ([18], § 4.2, Ex. 14) shows $J(R)/Ra = 0$, i.e. $J(R) = Ra$. Using the same method we get $(J(R))^n = Ra^n$, and consequently $(J(R))^n/(J(R))^{n+1} = Ra^n/Ra^{n+1} \cong R/(Ra^{n+1} : a^n)$. Further, $(Ra^2 : a) \subseteq (Ra : a^n)$, as one may check easily, and hence $(Ra^{n+1} : a^n)$ is a maximal left ideal since $(Ra^2 : a)$ is so and we are through.

4.2. Lemma (see [3]). *Let R be a local ring with nilpotent Jacobson radical $J(R)$. Then:*

(i) *If $J(R) = Ra$ for some $a \in R$, then every left ideal of R is of the form Ra^n, for suitable $n = 0, 1, 2, \dots$.*

(ii) *If $J(R) = Ra = bR$ for some $a, b \in R$, then every left right ideal of R is two-sided and of the form $Ra^n = a^n R = b^n R = Rb^n$ for suitable $n = 0, 1, 2, \dots$.*

Proof. (i) Let $I \subseteq R$ be a left ideal. We can clearly suppose $I \neq 0$, $I \neq R$ so that there is $0 \neq x \in I$. Since $I \subseteq J(R)$, $x = r_1 a$ for some $r_1 \in R$. If r_1 is not an invertible element of R, then $r_1 = r_2 a$. Repeating this argument we can find an invertible element $s \in R$ and a natural integer $l(x)$ with the property $x = sa^{l(x)}$, $l(x) < h$, the nilpotence degree of a. Setting $n=\min\{l(x),\ x\in I,\ x\neq 0\}$ we get $Ra^n\subseteq I$ (by the invertibility of s). On the other hand, for $x\in I$ we have $l(x)\geqq n$ and consequently $Ra^n=I$.

(ii) Let k be the nilpotence degree of $J(R)$ and let $a^n=0$. For $x_1, \dots, x_n\in J(R)$ we have $x_1=r_1a, \dots, x_n=r_na$ and $x_1\cdot\dots\cdot x_n=r_1a\cdot\dots\cdot r_na=r_1a\cdot\dots\cdot r'_{n-1}a^2=\dots=$ $=r'_na^n=0$ and hence $k=n$. If $c\in J(R)$ is of the nilpotence degree k, then $Rc=Ra^l$ for some natural integer l by (i) and the assumption $l>1$ leads to a contradiction with the nilpotence degree of c. Hence $Rc=Ra$. By the left-right symmetry we get $Ra=Rb=bR=aR$ and now it suffices to use induction.

4.3. We shall say that a left ideal I of a ring R satisfies ondition (β) if there are $\lambda_1, \dots, \lambda_n\in R$ such that $R\lambda_i=\lambda_i\cdot R$ is a maximal left ideal of R and $I=R\lambda_1\cdot R\lambda_2\cdots R\lambda_n$.

4.4. Lemma. *Let R be a ring with zero divisors satisfying (β) for any non-zero principal left ideal. Then the zero ideal satisfies (β), too.*

Proof. Let $\mathfrak{C}=\{\varrho\,|\,0\neq\varrho\in R,\ R\varrho=\varrho R\}$. Firstly suppose that $(0:\varrho)=0$ for each $\varrho\in\mathfrak{C}$. Then $(0:\lambda)=0$ for every $0\neq\lambda\in R$. Indeed, if $0\neq\lambda\in R$ then (by (β)) there is $\varrho\in\mathfrak{C}$ such that $R\lambda=R\varrho$. So $R/(0:\lambda)\cong R\lambda=R\varrho\cong R/(0:\varrho)=R$. However, any left ideal of R is a two-sided one, and hence $(0:\lambda)=(0: R/(0:\lambda))=(0:R)=0$ — a contradiction. Thus there is $\varrho\in\mathfrak{C}$ with $(0:\varrho)\neq 0$. Taking $\lambda\in(0:\varrho)$ non-zero we get $R\lambda\cdot R\varrho=$ $=R\lambda\cdot\varrho R=0$ and now it suffices to use (β) for $R\lambda$, $R\varrho$.

4.5. Lemma. *Let a ring R satisfy the condition of the preceding lemma. Then R posseses a finite left composition series and has therefore a finite number of maximal left ideals.*

Proof. By 4.4 we have $I_1\cdot\dots\cdot I_n=0$, where I_j are certain maximal and principal left ideals of R. We shall continue by induction on n. For $n=1$ there is nothing to prove. Let us assume our assertion holds for all $k<n$ and $I_1\cdot\dots\cdot I_n=0$ is an irredundant representation of 0 as a product of maximal and principal left ideals. Hence $I_1=(0: I_2\cdot\dots\cdot I_n)$ and so $I_2\cdot\dots\cdot I_n$ R/I_1 (because of the principality of $I_2, \dots, I_n$). Now it suffices to use the induction hypothesis for the ring $R/(I_2\cdot\dots\cdot I_n)$. The second part follows easily from the well-known theorems on composition series and from the following simple fact: If $R/K_1\cong R/K_2$ as modules for some two-sided ideals K_1, K_2 of R then $K_1=K_2$.

4.6. Theorem. *A ring R every nontrivial left ideal of which satisfies the condition (β), is stable.*

Proof. Let R be a ring without zero divisors. Then, due to (β), any left ideal of R is two-sided and a principal one and the condition from 3.4 holds trivially. Since every non-zero left ideal of R is isomorphic to R, R is left hereditary and we may use Theorem 3.5. The case of the ring with zero divisors is settled to the following more general result.

4.7. Theorem. *Let R be a ring with zero divisors. Then the following are equivalent:*

(i) *Every left ideal of R satisfies (β).*

(ii) *Every nontrivial principal left ideal of R satisfies (β).*

(iii) *R is a finite direct sum of local left and right principal ideal artinian rings. In this case, R is stable on the left and on the right.*

PROOF. (i) implies (ii) trivially.

(ii) implies (iii). By Lemma 4.5 R is artinian and $R/J(R)$ is a finite direct sum of division rings (since any left ideal of R is two-sided). It is a well-known fact (see e.g. [18]) that in this case idempotents can be lifted modulo $J(R)$. Hence $R=Re_1\oplus \oplus\cdots\oplus Re_n$, $e_i\in R$ are primitive orthogonal idempotents. However, Re_i are two-sided ideals and so $R=Re_1\dotplus\cdots\dotplus Re_n$. The rings Re_i are local and satisfy (β), thus it suffices to use 4.2.

(iii) implies (i). Let $R=R_1\dotplus\cdots\dotplus R_n$, R_i being local principal left and right ideal artinian rings. By 4.2 the rings R_i satisfy (β) for every left ideal and consequently every left ideal of R satisfies (β), too.

Now we shall prove the stability of R. With respect to 2.5 and (iii) we may assume R is a local artinian ring. By a result of GARDNER [8] such a ring has the trivial torsion theories only and R is therefore stable.

5. The subcommutative case

5.1. Theorem. *Every subcommutative principal left ideal ring R is stable.*

PROOF. We shall divide the proof into three steps:

(α) Suppose R is a prime ring. In this case R is without zero divisors and the subcommutativity of R permits us to prove, similarly as in the commutative case, that every non-zero left ideal of R satisfies (β). Now we can use 4.6.

(β) Suppose R is a local artinian ring. Then R is stable as follows from [8].

(γ) By using of GOLDIE's Theorem (see e.g. [12], Theorem 4.8 and Lemmas 4.12, 4.13) we may reduce the general situation to the above one's. Indeed, under our hypothesis, the ring R is a finite direct sum of primary rings and of local artinian rings.

5.2. H. BASS in [1] has introduced the notion of a perfect ring. He has shown that a right perfect ring can be defined as a ring satisfying the minimum condition on principal left ideals and that a commutative perfect ring is a finite direct sum of perfect local rings. This fact clearly holds for the subcommutative rings.

5.3. Theorem. *Any subcommutative perfect ring R is stable.*

PROOF. By the above remark and by 2.5 we can suppose R is a local perfect ring. But in this case R posseses the trivial torsion theories only (see GARDNER [8]) and it is therefore stable.

5.4. Theorem. *Any subcommutative noetherian ring R is h-stable.*

PROOF. We shall use the criterion 2.1(v). So let $\mathfrak{F}$ be a radical filter, $K\in\mathfrak{F}^*$, $I\subseteqq K$ an ideal and $(I:\lambda)\in\mathfrak{F}$ for each $\lambda\in K$. Since R is noetherian, every ideal posseses an (irredundant) representation as a finite intersection of meet irreducible ideals. Also we have $I=L_1\cap\cdots\cap L_n\cap L_{n+1}\cap\cdots\cap L_m$ and $K=L_1\cap\cdots\cap L_n$. We may restrict ourselves to the case $m>n$ since $m=n$ yields $I=K$. Now we are going to show $L_{n+1},\ldots,L_m\in\mathfrak{F}$. Denoting by P/L_{n+1} the prime radical of the ring R/L_{n+1} we get $(L_{n+1}:\varkappa)\in P$ for every $\varkappa\notin L_{n+1}$ since every zero divisor of R/L_{n+1} is contained in P/L_{n+1} (as assures us the irreducibility of L_{n+1} and the ascending chain condition for right annihilators). Further, taking $\varkappa \in K \dot{-} L_{n+1}$ we have $(I: \varkappa) \subseteq (L_{n+1}: \varkappa) \subseteq P$ and $P \in \mathfrak{F}$ since $(I: \varkappa) \in \mathfrak{F}$. However, the prime radical of a noetherian ring is nilpotent and hence $P^l \subseteq L_{n+1}$ for some natural integer l. Thus $L_{n+1} \in \mathfrak{F}$, similarly $L_{n+2}, \ldots, L_m \in \mathfrak{F}$ and finally $L_{n+1} \cap \cdots \cap L_m \in \mathfrak{F}$ which finishes the proof.

5.5. Proposition. Let R be a h-stable ring and $I \subseteq R$ be a two-sided ideal. Then R/I is h-stable.

Proof. We shall use the criterion 2.1 (v). Let $\mathfrak{E}$ be a radical filter of left ideals for R/I and let $f: R \to R/I$ denote the canonical projection. By [15] there exists a radical filter $\mathfrak{F}$ for R satisfying the following conditions:

(i) If $K \in \mathfrak{F}$, then $f(K) \in \mathfrak{E}$.

(ii) If $L \subseteq R$ is a left ideal with $I \subseteq L$ and $L/I \in \mathfrak{E}$, then $L \in \mathfrak{F}$.

Let $K \subseteq L$ be left ideal in R such that $I \subseteq K$, $(K/I: f(\lambda)) \in \mathfrak{E}$ for all $\lambda \in L$ and $L/I \in \mathfrak{E}^*$. For $\lambda \in L$ we have $I \subseteq (K: \lambda)$ and $f(K: \lambda) = (K/I: f(\lambda)) \in \mathfrak{E}$. Hence $(K: \lambda) \in \mathfrak{F}$ for all $\lambda \in L$. Further, let $(L: \alpha) \in \mathfrak{F}$ for some $\alpha \in R$. Then $f(L: \alpha) \in \mathfrak{E}$, i.e. $(L/I: f(\alpha)) \in \mathfrak{E}$. However, $L/I \in \mathfrak{E}^*$, and consequently $\alpha \in L$. From this we see that $L \in \mathfrak{F}^*$. By 2.1 (v) there exists a left ideal $A \subseteq R$ such that $K = A \cap L$ and $A \in \mathfrak{F}$. Then $K/I = A/I \cap L/I$ and $A/I \in \mathfrak{E}$.

5.6. Lemma. *Let R be a h-stable ring and $L, I \subseteq R$ be left ideals such that I is a two-sided ideal finitely generated as a left ideal and $IL = 0$. Then there is $n \geqq 1$ such that $I^n \cap L = 0$.*

Proof. Put $\mathfrak{E} = \{K \mid K$ is a left ideal and there is $n \geqq 1$ with $I^n \subseteq K\}$. By Proposition 0.6 of [20], $\mathfrak{E}$ is a radical filter. Let r be the corresponding radical. Since $IL = 0$, $(0: \lambda) \in \mathfrak{E}$ for all $\lambda \in L$, and so $L \subseteq r(R)$. With respect to 2.1 (v) there is a left ideal $A \in \mathfrak{E}$ such that $A \cap r(R) = 0$. In this case, there exists $n \geqq 1$ such that $I^n \cap L = 0$.

5.7. Lemma. *Let $I \subseteq R$ be a two-sided ideal such that I is finitely generated and maximal as a left ideal and I/I^2 is a simple left module. Let there exist a left ideal L and $n \geqq 0$ such that $L \subseteq I^n$ (here $I^0 = R$), $L \nsubseteq I^{n+1}$ and $IL = 0$. Then $I^{n+1} = I^{n+2}$.*

Proof. From 4.1 it follows that I^n/I^{n+1} is a simple module. Hence $I^n = I^{n+1} + L$, and consequently $I^{n+1} = I^{n+2} + IL = I^{n+2}$.

5.8. Lemma. *Let R be a h-stable ring and $I \subseteq R$ be a two-sided ideal such that I is finitely generated and maximal as a left ideal and I/I^2 is simple module. If $(0: I)_r = \{a \mid a \in R,\ Ia = 0\} \neq 0$, then there is $n \geqq 1$ such that $I^n = I^{n+1}$.*

Proof. Put $L = (0: I)_r$. Obviously L is an ideal and $IL = 0$. By 5.6 there exists $n \geqq 1$ such that $I^n \cap L = 0$. Since $L = 0$, $L \nsubseteq I^n$. Now it suffices to use 5.7.

5.9. Lemma. *Let R be a h-stable ring, $P \subseteq R$ a prime ideal which is not maximal as a left ideal and and $I \subseteq R$ be a finitely generated maximal left ideal such that I is two-sided and I/I^2 is simple. Then $IP = P$, provided $P^2 \subseteq IP$.*

Proof. Suppose $IP = P$. We have $P^2 \subseteq IP \subseteq P$. Put $\bar{R} = R/IP$ and denote by f the canonical projection of R onto $\bar{R}$. Then $f(P)$ is a prime ideal in $\bar{R}$, $f(I)$ is a maximal left ideal which is finitely generated and two-sided and $(f(P))^2 = 0$, $f(I).\ f(P) = 0$. From this one can easily deduce that $f(P)$ is the smallest prime ideal of $\bar{R}$. In particular, $\bar{R}$ is directly indecomposable as a ring. By 5.8 there exists $n \geqq 1$ such that

$(f(I))^n$ is idempotent. However, every idempotent ideal finitely generated as a left ideal in a h-stable ring is a ring direct summand (see [14]). So $f(I)=\bar{R}$ or $(f(I))^n=0$. In the first case, $I=R$ and hence $IP=P$, a contradiction. In the second case, $I^n\subseteqq IP\subseteqq P$ yields $I=P$, which contradicts the hypothesis.

5.10. Theorem. *Let R be a noetherian h-stable ring such that every maximal left ideal I of R is two-sided and I/I^2 is a simple left module. Then R is a finite direct sum of prime rings and of local artinian left principal ideal rings.*

PROOF. Since R is noetherian, R is a finite direct sum of directly indecomposable rings. Thus it suffices to assume R is directly indecomposable. We shall distinguish wo cases:

(i) Every prime ideal of R is a maximal left ideal. By Theorem 3.6. of [16] R is artinian. R has no nontrivial idempotent ideals by [14] so that R has primary decompositions by Lemma 3.3 of [3] and consequently R is local owing to the indecomposability and the hypothesis. By 4.1 and 4.2, R is left principal ideal ring.

(ii) R contains a prime ideal which is not a maximal left ideal. Assuming $P^2\neq P$ let us denote by $\bar{R}$ the factor-ring R/P^2 and by f the canonical projection of R onto $\bar{R}$. Since $\bar{R}$ is left noetherian, there is a maximal ideal $I\subseteqq R$ with $f(I)f(P)\neq fP$ which contradicts to 5.9. Thus $P^2=P$, P is a ring direct summand of R by [14] and consequently $P=0$ owing to the indecomposability of R. Thus R is prime and the proof is finished.

A ring R will be called a dedekind ring if every ideal of R is a finite product of prime ideals.

5.11. Corollary. Let R be a subcommutative dedekind ring in which prime ideals commute. Then R is a finite direct sum of dedekind prime rings and local artinian principal ideal rings. Moreover, R is noetherian and stable.

PROOF. Let $P\subseteqq R$ be a prime ideal. Then R/P is a dedekind prime ring. Using the same method as in [17] we may prove that every non-zero prime ideal of R/P is a maximal ideal. From this it follows that every prime ideal of R is either a minimal prime ideal or a maximal ideal. This fact and the dedekind property yield the ascending chain condition, as one may check easily. Further, for every maximal ideal I, I/I^2 is a simple module. From 5.4 and 5.10 it follows that R is a finite direct sum of dedekind prime rings and of local artinian principal ideal rings. By Theorem 4.4 of [23] every dedekind prime ring is hereditary and the stability of R follows now from 3.5, 5.3 and 2.5.

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