Convolution quotients and distributions

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Introduction

In this paper, using a slight modification of a quotient module construction [7], we generalize the concept of Mikusiński operators and establish some intimate connections between Schwartz distributions and convolution quotients introduced here.

This generalization subsumes the cases of operators associated with admissible convolution rings [3, 4], and may be considered as a general foundation for the convolution calculus.

The identification, used here to identify certain distributions as convolution quotients, is more general than the traditional one [1].

§ 1. A universal module of convolution quotients

Let $\mathscr{E}$ be the set of all infinitely differentiable functions from the k-dimensional Euclidean space $\mathbf{R}^k$ into the field $\mathbf{C}$ of complex numbers, and let $\mathscr{D}$ be the subset of $\mathscr{E}$ consisting of all those functions with compact support.

It is known, that under addition and convolution $\mathscr{D}$ is a commutative ring without proper zero-divisors, and $\mathscr{E}$ is a $\mathscr{D}$-module.

Definition 1.1. Let $\mathscr{D}\subset R\subset M\subset\mathscr{E}$ such that under addition and convolution R is a ring and M is an R-module.

Remark 1.2. Because of the commutativity of convolution, we have $\varphi*f=f*\varphi$ for all $\varphi\in R$ and $f\in M$.

Example 1.3. $M=R=\mathscr{D}$.

Example 1.4. $M=R=\mathscr{E}_R$, where $\mathscr{E}_R$ denotes the subset of $\mathscr{E}$ consisting of all those functions with supports contained in various right-sided orthants [3].

Example 1.5. $M=R=\mathscr{E}_L$, where $\mathscr{E}_L$ denotes the subset of $\mathscr{E}$ consisting of all those functions with supports contained in various left-sided orthants [4].

Example 1.6. $M=R=\mathscr{E}_{\exp}$, where $\mathscr{E}_{\exp}$ denotes the subset of $\mathscr{E}$ consisting of all those functions of exponential descent [3].

Example 1.7. $M=R=\mathcal{S}$, where $\mathcal{S}$ denotes the set of all rapidly decreasing functions on $\mathbf{R}^k$ [6].

Example 1.8. $M=R=L^1\cap\mathcal{E}$, where L^1 denotes the set of all integrable functions on $\mathbf{R}^k$ [8].

Example 1.9. $M=\mathcal{E}$ and $R=\mathcal{D}$.

Definition 1.10. Let S be a multiplicative system [7] in R, i.e., $\emptyset\neq S\subset R$, $0\notin S$ and $S*S\subset S$.

Definition 1.11. For any $(\varphi, f)\in S\times M$ let

$$\frac{f}{\varphi} = \{(\psi, g)\in S\times M : \exists\sigma\in S : (f*\psi-\varphi*g)*\sigma = 0\}.$$

Let

$$\mathcal{M}(M, R, S) = \left\{\frac{f}{\varphi} : (\varphi, f)\in S\times M\right\}$$

and

$$\mathcal{R}(M, R, S) = \left\{\frac{f}{\varphi}\in\mathcal{M}(M, R, S) : \exists(\varphi_1, f_1)\in\frac{f}{\varphi} : f_1\in R\right\}.$$

For any $\dfrac{f}{\varphi}, \dfrac{g}{\psi}\in\mathcal{M}(M, R, S)$ let

$$\frac{f}{\varphi}+\frac{g}{\psi} = \frac{f*\psi+\varphi*g}{\varphi*\psi}.$$

For any $\dfrac{f}{\varphi}\in\mathcal{R}(M, R, S)$ and $\dfrac{g}{\psi}\in\mathcal{M}(M, R, S)$ let

$$\frac{f}{\varphi}*\frac{g}{\psi} = \frac{f_1*g}{\varphi_1*\psi} \quad\text{and}\quad \frac{g}{\psi}*\frac{f}{\varphi} = \frac{g*f_1}{\psi*\varphi_1},$$

where $(\varphi_1, f_1)\in\dfrac{f}{\varphi}$ such that $f_1\in R$.

Theorem 1.12. *$\mathcal{R}(M, R, S)$ is a commutative ring with unity, and $\mathcal{M}(M, R, S)$ is a unitial $\mathcal{R}(M, R, S)$-module.*

To prove this use [7]. For example, we show that the definition of $\dfrac{f}{\varphi}*\dfrac{g}{\psi}$ is correct. For this suppose that $(\varphi_1, f_1), (\varphi_2, f_2)\in\dfrac{f}{\varphi}$ such that $f_1, f_2\in R$ and $(\chi, h)\in\dfrac{g}{\psi}$. Then there exist $\sigma_1, \sigma_2\in S$ such that

$$(f_1*\varphi_2-\varphi_1*f_2)*\sigma_1 = 0 \quad\text{and}\quad (g*\chi-\psi*h)*\sigma_2 = 0.$$

Hence, it follows that

$$((f_1 * g) * (\varphi_2 * \chi)) * (\sigma_1 * \sigma_2) = ((g * f_2) * (\varphi_1 * \chi)) * (\sigma_1 * \sigma_2)$$

and

$$((g * f_2) * (\varphi_1 * \chi)) * (\sigma_1 * \sigma_2) = ((\varphi_1 * \psi) * (f_2 * h)) * (\sigma_1 * \sigma_2).$$

Thus

$$((f_1 * g) * (\varphi_2 * \chi) - (\varphi_1 * \psi) * (f_2 * g)) * (\sigma_1 * \sigma_2) = 0,$$

i. e.,

$$\frac{f_1 * g}{\varphi_1 * \psi} = \frac{f_2 * h}{\varphi_2 * \chi}.$$

Finally, observe that if $\frac{g}{\psi} \in \mathscr{R}(M, R, S)$ also holds then the two definitions for $\frac{f}{\varphi} * \frac{g}{\psi}$ given the same result.

Example 1.13. $\mathscr{M}(\mathscr{C}_R, \mathscr{C}_R, \mathscr{C}_R \setminus \{0\})$ is the field of Mikusiński operators.

Example 1.14. $\mathscr{M}(\mathscr{E}, \mathscr{D}, \mathscr{D} \setminus \{0\})$ is a vector space over $\mathscr{R}(\mathscr{E}, \mathscr{D}, \mathscr{D} \setminus \{0\})$.

§ 2. Embedding of certain distributions

Let $\mathscr{D}'$ be the set of all Schwartz distributions on $\mathbf{R}^k$, and let $\mathscr{E}'$ be the subset of $\mathscr{D}'$ consisting of all those distributions with compact support.

It is known that under addition and convolution $\mathscr{E}'$ is a commutative ring with unity and without proper zero-divisors, and $\mathscr{D}'$ is a unitial $\mathscr{E}'$-module such that $\mathbf{C}$ and $\mathscr{E}$ are embedded in $\mathscr{D}'$ such that $\mathbf{C}, \mathscr{D} \subset \mathscr{E}'$.

Hereafter, S is supposed to have non-empty intersection with $\mathscr{D}$.

Theorem 2.1. *Let $\varphi \in S \cap \mathscr{D}$. Then the mapping defined on $\mathscr{E}'$ by*

$$\Lambda \to \frac{\Lambda * \varphi}{\varphi}$$

is independent of φ and is a ring isomorphism of $\mathscr{E}'$ into $\mathscr{R}(M, R, S)$.

For a proof see the proofs of Theorem 2.8. and 2.14.

Definition 2.2. For any $\Lambda \in \mathscr{E}'$ let

$$\Lambda = \frac{\Lambda * \varphi}{\varphi}$$

where $\varphi \in S \cap \mathscr{D}$.

Remark 2.3. After this embedding $\mathscr{M}(M, R, S)$ may also be considered as an $\mathscr{E}'$-module.

Theorem 2.4. *Let $q \in \mathscr{M}(M, R, S)$. Then $q \in \mathscr{E}'$ if and only if $q * \mathscr{D} \subset \mathscr{D}$.*

For a proof see the proof of Theorem 2.16.

Example 2.5. Let $\varphi\in S\cap\mathcal{D}$. Then

$$\frac{D^\alpha\varphi}{\varphi}\quad\text{and}\quad\frac{\mathcal{T}_\lambda\varphi}{\varphi}$$

belong to $\mathcal{E}'$ for any k-tuple α of nonnegative integers and $\lambda\in\mathbf{R}^k$ [6].

Definition 2.6. Let $\mathcal{E}'\subset\mathcal{N}\subset\mathcal{D}'$ be a submodule of the $\mathcal{E}'$-module $\mathcal{D}'$ such that $\mathcal{N}*(S\cap\mathcal{D})\subset M$.

Example 2.7. $\mathcal{N}=\{\Lambda\in\mathcal{D}':\Lambda*\mathcal{D}\subset M\}$.

Theorem 2.8. *Let $\varphi\in S\cap\mathcal{D}$. Then the mapping Φ defined on $\mathcal{N}$ by*

$$\Phi(\Lambda)=\frac{\Lambda*\varphi}{\varphi}$$

is independent of φ and is an $\mathcal{E}'$-module homomorphism of $\mathcal{N}$ into $\mathcal{M}(M, R, S)$.

Proof. Suppose that $\Lambda\in\mathcal{N}$ and $\varphi_1,\varphi_2\in S\cap\mathcal{D}$. Then $(\Lambda*\varphi_1)*\varphi_2=\varphi_1*(\Lambda*\varphi_2)$. Hence $\big((\Lambda*\varphi_1)*\varphi_2-\varphi_1*(\Lambda*\varphi)\big)*\sigma=0$ for any $\sigma\in S$. Thus

$$\frac{\Lambda*\varphi_1}{\varphi_1}=\frac{\Lambda*\varphi_2}{\varphi_2}.$$

Suppose now that $\Lambda_1, \Lambda_2\in\mathcal{N}$. Then

$$\Phi(\Lambda_1+\Lambda_2)=\frac{(\Lambda_1+\Lambda_2)*\varphi}{\varphi}=\frac{(\Lambda_1+\Lambda_2)*(\varphi*\varphi)}{\varphi*\varphi}=\frac{(\Lambda_1*\varphi)*\varphi+\varphi*(\Lambda*\varphi)}{\varphi*\varphi}$$

$$=\frac{\Lambda_1*\varphi}{\varphi}+\frac{\Lambda_2*\varphi}{\varphi}=\Phi(\Lambda_1)+\Phi(\Lambda_2),$$

and if in addition $\Lambda_1\in\mathcal{E}'$ then

$$\Phi(\Lambda_1*\Lambda_2)=\frac{(\Lambda_1*\Lambda_2)*\varphi}{\varphi}=\frac{(\Lambda_1*\Lambda_2)*(\varphi*\varphi)}{\varphi*\varphi}=\frac{(\Lambda_1*\varphi)*(\Lambda_2*\varphi)}{\varphi*\varphi}=$$

$$=\frac{\Lambda_1*\varphi}{\varphi}*\frac{\Lambda_2*\varphi}{\varphi}=\Lambda_1*\Phi(\Lambda_2).$$

Example 2.9. Let $M=\mathcal{E}$, $R=\mathcal{D}$, $S=\mathcal{D}\setminus\{0\}$ and $\mathcal{N}=\mathcal{D}'$. Then Φ fails to be one-to-one.

Namely, for example

$$\Phi(f)=\frac{f*\varphi}{\varphi}=0$$

for all constant $f\in\mathcal{E}$. To see this choose $\varphi\in S$ such that $\int_{\mathbf{R}^k}\varphi=0$.

Definition 2.10. $\mathcal{N}$ is said to be normal with respect to (M, R, S) if $\Lambda\in\mathcal{N}$, $\varphi\in S\cap\mathcal{D}$, $\sigma\in S$ and $(\Lambda*\varphi)*\sigma=0$ imply that $\Lambda=0$.

Remark 2.11. If $S\subset\mathcal{D}$ then $\mathcal{N}$ is normal with respect to (M, R, S) if and only if $\Lambda\in\mathcal{N}$, $\varphi\in S$ and $\Lambda*\varphi=0$ imply that $\Lambda=0$.

Remark 2.12. If $f\in M$, $\varphi\in S$ and $f*\varphi=0$ imply that $f=0$, and $\mathcal{N}*\mathcal{D}\subset M$, then $\mathcal{N}$ is normal with respect to (M, R, S).

Namely, suppose that $\Lambda\in\mathcal{N}$, $\varphi\in S\cap\mathcal{D}$, $\sigma\in S$ such that $(\Lambda*\varphi)*\sigma=0$. Then $(\Lambda*\psi)*(\varphi*\sigma)=0$ for all $\psi\in\mathcal{D}$. Hence, it follows that $\Lambda*\psi=0$ for all $\psi\in\mathcal{D}$. This implies that $\Lambda=0$.

Example 2.13. Let $M\in\{\mathcal{E}_R, \mathcal{E}_L, \mathcal{E}_{\exp}\}$, $R=M$, $S=M\setminus\{0\}$ and $\mathcal{N}$ as in Example 11. Then $\mathcal{N}$ is normal with respect to (M, R, S).

To prove this use Remark 2.12. and the fact that M has no proper zero-divisors [3].

Theorem 2.14. *Φ is one-to-one if and only if $\mathcal{N}$ is normal with respect to (M, R, S).*

Proof. First let Φ be one-to-one. Suppose that $\Lambda\in\mathcal{N}$, $\varphi\in S\cap\mathcal{D}$ and $\sigma\in S$ such that $(\Lambda*\varphi)*\sigma=0$. Then

$$\big((\Lambda*\varphi)*\varphi-\varphi*(0*\varphi)\big)*\sigma=0.$$

Hence, it follows that

$$\frac{\Lambda*\varphi}{\varphi}=\frac{0*\varphi}{\varphi}.$$

Thus $\Phi(\Lambda)=\Phi(0)$, and so $\Lambda=0$.

Now let $\mathcal{N}$ be normal with respect to (M, R, S). Suppose that $\Lambda_1, \Lambda_2\in\mathcal{N}$ such that $\Phi(\Lambda_1)=\Phi(\Lambda_2)$. Then

$$\frac{\Lambda_1*\varphi}{\varphi}=\frac{\Lambda_2*\varphi}{\varphi}$$

where $\varphi\in S\cap\mathcal{D}$. Hence, it follows that there exists $\sigma\in S$ depending on φ such that

$$\big((\Lambda_1*\varphi)*\varphi-\varphi*(\Lambda_2*\varphi)\big)*\sigma=0.$$

Thus

$$\big((\Lambda_1-\Lambda_2)*\varphi\big)*(\varphi*\sigma)=0,$$

and so $\Lambda_1=\Lambda_2$.

Definition 2.15. If $\mathcal{N}$ is normal with respect to (M, R, S) then for any $\Lambda\in\mathcal{N}$ let

$$\Lambda=\frac{\Lambda*\varphi}{\varphi},$$

where $\varphi\in S\cap\mathcal{D}$.

Theorem 2.16. *Suppose that $\mathcal{N}$ is normal with respect to (M, R, S). Let $q\in\mathcal{M}(M, R, S)$ and suppose that $q*\mathcal{D}\subset\mathcal{E}$. Then there exists a $\Lambda\in\mathcal{D}'$ such that*

$$q=\frac{\Lambda*\varphi}{\varphi}$$

for any $\varphi\in S\cap\mathcal{D}$.

PROOF. Let F be the function defined on $\mathscr{D}$ by

$$F(\varphi) = q * \varphi.$$

Then

$$F(\varphi * \psi) = F(\varphi) * \psi$$

for all $\varphi, \psi \in \mathscr{D}$. Thus, by Lemma 3. in [2], there exists a unique $\Lambda \in \mathscr{D}'$ such that

$$F(\varphi) = q * \varphi = \Lambda * \varphi$$

for all $\varphi \in \mathscr{D}$. Hence, it follows that

$$q = \frac{\Lambda * \varphi}{\varphi}$$

for any $\varphi \in S \cap \mathscr{D}$.

Note. A preliminary version of this paper with title "Convolution quotients of a new type and their connection with distributions" was presented at the Conference on Generalised Functions in Wisła, Poland, October 1973. Concerning it C. RYLL—NARDZEWSKI at the Conference proved that if $k=1$ then for every $\varphi \in \mathscr{D}$ there exists $z \in \mathbf{C}$ such that $\{e^{zt}\} * \varphi = 0$. This shows that for $k=1$ there is no multiplicative system S in $\mathscr{D}$ so that $\mathscr{D}'$ should be normal with respect to $(\mathscr{E}, \mathscr{D}, S)$.

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