# Conditional-completion of Boolean rings by lower cuts

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## 1. Introduction

In what follows $(B, \leqq)$ denotes a Boolean ring *not necessarily with a unit* (cf. [1, p. 130]). Hence, $(B, \leqq)$ is not necessarily a Boolean algebra. Accordingly, $(B, \leqq)$ is a distributive lattice with the least element 0 and $B$ is closed under subtraction. This means that $(B, \leqq)$ is a partially ordered set such that for every element $x$ and $y$ of $B$, the least upper bound $x \vee y$ as well as the greatest lower bound $x \wedge y$ exists, and for every element $z$ of $B$

$$z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) \tag{1}$$

which implies that for every element $x$, $y$, $z$ of $B$

$$z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y). \tag{2}$$

Moreover, for every element $x$ and $y$ of $B$, the *difference* $x - y$ exists, where $d = x - y$ is the unique element of $B$ such that

$$d \vee y = x \vee y \quad \textit{and} \quad d \wedge y = 0. \tag{3}$$

As usual, a Boolean ring $(B, \leqq)$ is called *conditionally complete* if and only if every nonempty subset $S$ of $B$ (which, a priori, is bounded below by 0) has a greatest lower bound *(infimum)* denoted by $\wedge S$ or by $\bigwedge_{x \in S} x$. Equivalently, a Boolean ring $(B, \leqq)$ is conditionally complete, if and only if every subset $E$ of $B$ which is bounded above has a least upper bound *(supremum)* denoted by $\vee E$ or by $\bigvee_{x \in E} x$. Clearly, in $(B, \leqq)$ we have $\vee \emptyset = 0$.

Obviously, a Boolean ring $(B, \leqq)$ need not be conditionally complete. However, as shown in this paper for every Boolean ring $(B, \leqq)$ there exists a conditionally complete Boolean ring $(L, \subseteq)$ (where $L$ is the set of all the lower cuts of $(B, \leqq)$, see Definition 1 below such that there exists an isomorphism (i.e., a one-to-one mapping which preserves order in both directions) $f$ from $(B, \leqq)$ into $(L, \subseteq)$ which is also *infima and suprema preserving*. This means that the isomorphism $f$ is such that for every subset $S$ of $B$

$$f\left(\bigwedge_{x \in S} x\right) = \bigwedge_{x \in S} f(x) \tag{4}$$

whenever the left side of the above equality exists. Also, for every subset $E$ of $B$

$$f(\bigvee_{x\in S} x) = \bigvee_{x\in E} f(x) \tag{5}$$

whenever the left side of the above equality exists. Clearly, in (4) and (5) the left side of each equality refers to $(B, \leqq)$ whereas the right side of each equality refers to $(L, \subseteqq)$.

It is customary to refer to an isomorphism $f$ described above or to the Boolean ring $(L, \subseteqq)$ described above, as a *conditional-completion* of the Boolean ring $(B, \leqq)$.

As shown below, the conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ that we consider in this paper, has the following special features. The lattice-theoretic order in $(L, \subseteqq)$ is the set-theoretic inclusion $\subseteqq$. Moreover, for every element $X$ and $Y$ of $(L, \subseteqq)$, the infimum $X\wedge Y$ is the set-theoretical intersection $X\cap Y$. However, for every element $X$ and $Y$ of $(L, \subseteqq)$, the supremum $X\vee Y$ is generally larger (w.r.t. $\subseteqq$) than the set-theoretical union $X\cup Y$. Also, for every element $X$ and $Y$ of $(L, \subseteqq)$, the difference $X-Y$ is generally smaller (w.r.t. $\subseteqq$) than the set-theoretical difference $X\setminus Y$. In fact, we show below that the conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ is a conditionally-complete Boolean ring in which

$$\wedge G = \cap G \quad \textit{for every nonempty subset } G \textit{ of } L \tag{6}$$

whereas, as mentioned, $\vee H \supseteqq \cup H$ for every subset $H$ of $L$ which is bounded above and $X-Y\subseteqq X\setminus Y$ for every element $X$ and $Y$ of $(L, \subseteqq)$.

Our conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ resembles Dedekind's conditional-completion of rational numbers by real numbers, whereby every real number $r$ is identified with the set of all rational numbers $x$ such that $x\leqq r$.

Also, our conditional-completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ closely resembles MacNeille's original work [2] on completion by cuts of a Boolean algebra (i.e., a Boolean ring with unit). However, as the reader will note, between [2] and the present paper, there exist substantial differences in definitions, in methods and in proofs.

Conditional-completion of partially ordered sets and in particular conditional-completion of Boolean rings are of significant importance in mathetics. However, it seems that there is nothing in print which treats these subjects elegantly, lucidly and in a self-contained readable and understandable way without resorting to complicated topological considerations. It is hoped that the present paper will contribute to the subject of the conditional-completion of Boolean rings. The obvious relation of a completion of a partially ordered set (or a Boolean ring) to the conditional-completion of the partially ordered set (or the Boolean ring) is mentioned in the sequel.

## 2. Lower cuts and conditional-completion of partially ordered sets

Let $(P, \leqq)$ be a partially ordered set. For every element $x$ of $P$ we call the subset $I(x)$ of $P$ a *lower segment* of $P$ if and only if

$$I(x) = \{y \mid y\in P \textit{ and } y \leqq x\}. \tag{7}$$

Based on (7) we introduce:

Definition 1. *A subset $W$ of a partially ordered set $(P, \leqq)$ is called a lower cut of $P$ if and only if for some nonempty subset $S$ of $P$*

$$W = \bigcap_{x \in S} I(x). \tag{8}$$

In other words, a lower cut of a partially ordered set $P$ is the intersection of a nonempty family of lower segments of $P$.

Let $(P, \leqq)$ be a partially ordered set, then

$$\bigcap_{x \in P} I(x) = \{0\} \quad or \quad \bigcap_{x \in P} I(x) = \emptyset \tag{9}$$

where 0 is the minimum (the least) element of $P$. This is because if $\bigcap_{x \in P} I(x) \neq \emptyset$ then there exists an element $h \in \bigcap_{x \in P} I(x)$ and therefore by (7) we have $h \leqq x$ for every $x \in P$. Thus, $h$ is the (unique) minimum element of $P$.

**Lemma 1.** *Let $(P, \leqq)$ be a partially ordered set. Every nonempty subst of $P$ which is bounded below has an infimum if and only if every nonempty subset of $P$ which is bounded above has a supremum.*

PROOF. The proof of the lemma follows easily from the fact that for every subset $S$ of $(P, \leqq)$ we have:

$$\inf S = \sup \{x \mid x \textit{ is a lower bound of } S\}$$

and

$$\sup S = \inf \{x \mid x \textit{ is an upper bound of } S\}.$$

Definition 2. *A partially ordered set $P$ is called conditionally complete if and only if every nonempty subset of $P$ which is bounded below has a greatest lower bound (or every nonempty subset of $P$ which is bounded above has a least upper bound).*

Based on Definitions 1 and 2 we prove:

**Lemma 2.** *Let $L$ be the set of all lower cuts of a partially ordred set $(P, \leqq)$. Then $(L, \subseteqq)$ is a conditionally complete partially ordered set such that for every nonempty subset $G$ of $L$ which is bounded below*

$$\inf G = \wedge G = \cap G. \tag{10}$$

*Moreover, for every nonempty subset $H$ of $L$ which is bounded above*

$$\sup H = \vee H \supseteqq \cup H. \tag{11}$$

PROOF. Clearly, $(L, \subseteqq)$ is a partially ordered set since $\subseteqq$ is the set-theoretical inclusion. In order to prove that $(L, \subseteqq)$ is conditionally complete, in view of Lemma 1, it is sufficient to establish (10). Now, let $G$ be as described in the statement of the Lemma. But then by (8) we see that for some nonempty subset $K$ of the powerset (i.e., the set of all subset) of $P$

$$G = \Big\{W \mid W = \bigcap_{x \in S} I(x) \quad and \quad \emptyset \neq S \in K\Big\}.$$

But then

$$\cap G = \bigcap_{S \in K} \Big(\bigcap_{x \in S} I(x)\Big) = \bigcap_{x \in \cup K} I(x).$$

Thus, by (8), we see that $\cap G$ is a lower cut of $(P, \leqq)$ and therefore $\cap G \in L$. Hence, from Lemma 1 and Definition 2 it follows that $(L, \subseteqq)$ is conditionally complete. Moreover, since $\subseteqq$ is the set-theoretical inclusion, in view of Lemma 1 we see that (10) implies (11).

**Theorem 1.** *Let $(P, \leqq)$ be a partially ordered set and $(L, \subseteqq)$ be the conditionally complete partially ordered set of all lower cuts of $P$. Then the function $f$ given by*

$$f(x) = I(x) \tag{12}$$

*is a one-to-one suprema and infima preserving mapping from $(P, \leqq)$ into $(L, \subseteqq)$.*

PROOF. Let $x$ and $y$ be distinct elements of $P$. Without loss of generality, we may assume $x \not\leqq y$. But then by (7) and (12) we see that $x \in f(x)$ and $x \notin f(y)$. Hence $f(x) \neq f(y)$ and consequently, $f$ is one-to-one.

Next, we show that $f$ is order preserving, i.e., for every element $x$ and $y$ of $P$, we have:

$$x < y \text{ if and only if } f(x) \subset f(y). \tag{13}$$

Let $x < y$ and let $z \in f(x)$. By (7) and (12) we see that $z \leqq x < y$ and therefore $z \in f(y)$ which implies $f(x) \subset f(y)$. Conversely, let $f(x) \subset f(y)$. But then clearly by (7) and (12) we have $x < y$.

Now, we show that $f$ is infima preserving. In view of (13), it is enough to show that $f$ preserves the infimum of every nonempty subset of $P$. Let $S$ be a nonempty subset of $P$. By (12) we have:

$$f[S] = \{I(x) \mid x \in S\}.$$

On the other hand, from (10) it follows that

$$\inf f[S] = \wedge f[S] = \cap f[S] = \bigcap_{x \in S} I(x).$$

Let $s = \inf S = \wedge S$. But then, by (7) and (12) we have

$$f(s) = I(s) \subseteqq \bigcap_{x \in S} I(x).$$

On the other hand, if $y \in \bigcap_{x \in S} I(x)$ then clearly $y \leqq s$ since $s = \inf S$. Hence

$$\bigcap_{x \in S} I(x) \subseteqq \{y \mid y \leqq s\} = I(s).$$

Thus, $f(\wedge S) = \cap f[S]$ and $f$ is infima preserving. From this, in view of the two equalities used in the proof of Lemma 1 it follows that $f$ is also suprema preserving. As indicated by (11), obviously, $\vee f[E] \supseteqq \cup f[E]$.

Since $f$ is a one-to-one suprema and infima preserving mapping, it is customary to refer to $(L, \subseteqq)$ as the *conditional-completion by lower cuts of* $(P, \leqq)$. In $(L, \subseteqq)$, every element $x$ of $P$ is identified with the lower segment $I(x)$ of $P$.

*Remark* 1. A partially ordered set is called *complete* if and only if every subset of it has an infimum (or equivalently, has a supremum). As usual, by adjoining at most two elements to a conditional-completion of a partially ordered set $(P, \leqq)$, a *completion* of $(P, \leqq)$ can be obtained. Thus, if $(P, \leqq)$ has neither a least nor a greatest element, then by adjoining $\emptyset$ and $P$ respectively to $(L, \subseteqq)$, a completion of $(P, \leqq)$ is obtained.

## 3. Conditional-completion of Boolean rings

In this section we show that if instead of an arbitrary partially ordered set we start with a Boolean ring $(B, \leqq)$, then the conditional-completion $(L, \subseteqq)$ by lower cuts of $(B, \leqq)$ is again a Boolean ring.

We recall that a Boolean ring is a *distributive lattice which is closed under subtraction* (see (1), (2), (3)).

Let us recall also that a lattice $(M, \leqq)$ with least element 0 is called *sectionally complemented* [3, p. 28] *if and only if for every element $x$ and $u$ of $M$ such that $x \leqq u$, there exists an element $y$ of $M$ such that*

$$x \vee y = u \quad \textit{and} \quad x \wedge y = 0 \tag{14}$$

in which case $y$ is called *a complement of $x$ with respect to $u$*. It is easily seen that if $(M, \leqq)$ is a distributive lattice then an element $x$ of $M$ has at most one complement $y$ with respect to an element $u$ of $M$.

Although we define a Boolean ring as a distributive lattice which is closed under subtraction, as shown by the lemma below, we can also regard a Boolean ring as a sectionally complemented distributive lattice (and equivalently, [3, p. 48] or [1, p. 130] as a *relatively complemented distributive lattice with least element* 0).

**Lemma 3.** *A distributive lattice $(B, \leqq)$ is a Boolean ring if and only if $(B, \leqq)$ is sectionally complemented.*

Proof. Let $(B, \leqq)$ be a Boolean ring, and let $x$ and $u$ be elements of $B$ such that $x \leqq u$. Then clearly, the difference $u-x$, as given by (3), is the complement of $x$ with respect to $u$. Thus $(B, \leqq)$ is sectionally complemented.

Next, let $(B, \leqq)$ be a sectionally complemented distributive lattice with least element 0, and let $x$ and $y$ be elements of $B$. Let $d$ be the complement of $y$ with respect to $x \vee y$. Thus, $d \vee y = x \vee y$ and $d \wedge y = 0$, and therefore, in view of (3), we see that $d$ is the difference $x-y$. Hence, $(B, \leqq)$ is closed under subtraction and the lemma is proved.

As mentioned earlier, we show in this paper that the conditional completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ is a (conditionally complete) Boolean ring. To this end, in view of the above, it is enough to prove that the conditional completion $(L, \subseteqq)$ of a sectionally complemented distributive lattice $(B, \leqq)$ is again a sectionally complemented distributive lattice. In what follows, we shall prove this by a sequence of lemmas and partial results. Many of the lemmas are established to prove the distributivity of $(L, \subseteqq)$. It seems that no shorter and direct proof of the distributivity of $(L, \subseteqq)$ is plausible.

Notation. *In what follows, we let* $(L, \subseteq)$ *represent the conditional completion by lower cuts of a Boolean ring* $(B, \leqq)$.

**Lemma 4.** $(L, \subseteq)$ *is a lattice with least element* $\{0\}$.

Proof. Obviously $\{0\}$ is the least element of $(L, \subseteq)$ because 0 is an element of every lower cut of $B$. Thus to prove the lemma, it remains to show that if $X$ and $Y$ are elements of $L$, then $X$ and $Y$ have a supremum and an infimum in $L$.

Let $X$ and $Y$ be elements of $L$. Then $X$ and $Y$ can be given in the form $X = \bigcap_{v \in M} I(v)$ and $Y = \bigcap_{v \in N} I(v)$, and let $m \in M$ and $n \in N$. Since $B$ is a Boolean ring and $M \subseteq B$ and $N \subseteq B$, it is obvious that $(m \vee n) \in B$ and therefore by (12) we have $I(m \vee n) \in L$. But then $X \subseteq I(m \vee n)$ and $Y \subseteq I(m \vee n)$. Consequently $X$ and $Y$ are bounded above in $(L, \subseteq)$ by $I(m \vee n)$, and thus have a supremum in $(L, \subseteq)$ since $(L, \subseteq)$ is conditionally complete.

On the other hand, from (10) it is clear that $\bigcap_{v \in (M \cup N)} I(v)$ is the infimum $X \wedge Y = X \cap Y$ of $X$ and $Y$ in $(L, \subseteq)$.

Hence the lemma is proved.

**Lemma 5.** $(L, \subseteq)$ *is sectionally complemented.*

Proof. Let $X$ and $Y$ be elements of $L$ such that $X = \bigcap_{v \in M} I(v)$ where $M$ is the set of *all* upper bounds of $X$ and $Y = \bigcap_{v \in N} I(v)$ where $N$ is the set of *all* upper bounds of $Y$. Moreover let

(15) $$X \subseteq Y.$$

To prove the lemma, it is sufficient to show that the lower cut $D$ of $B$ given by

(16) $$D = \cap \{I(v) \mid v = (n - x) \text{ for some } n \in N \text{ and some } x \in X\}$$

is a complement of $X$ with respect to $Y$ in $(L, \subseteq)$. In the above, the difference $n - x$ exists since $B$ is a Boolean ring and $N \subseteq B$ and $X \subseteq B$.

In view of (14), we must show that

(17) $$X \cap D = \{0\}$$

and

(18) $$X \vee D = Y.$$

Let $x \in (X \cap D)$. Then $x \leqq (n - x)$ for some $n \in N$, which implies $x = 0$ since $B$ is a Boolean ring. Thus (17) is established.

We note that $D \subseteq Y$ because $v \in D$ implies $v \leqq (n - x)$ for every $n \in N$ and every $x \in X$. Hence $v \leqq n$ for every $n \in N$, and thus $v \in Y$. Consequently from (17) it follows that

(19) $$D \subseteq ((Y \setminus X) \cup \{0\}).$$

We claim that an element $u$ of $B$ is an upper bound of $X \cup D$ if and only if $u \in N$. IF $u \in N$ then $u$ is an upper bound of $Y$ and hence $u$ is an upper bound of $X \cup D$ since $(X \cup D) \subseteq Y$ as shown by (15) and (19).

Next, let $u$ be an upper bound of $X \cup D$ and let us assume, on the contrary, that $u \notin N$. Then since $u$ is an upper bound of $X$, we see that $u \in M$ and consequently by our assumpion, $u \in (M \setminus N)$. However, since $u \notin N$, there exists $y \in (Y \setminus X)$ such that

(20) $$y \nleqq u.$$

We show that $(y-u) \in D$. Since $u$ is an upper bound of $X$ we have $x \leqq u$ for every $x \in X$. Furthermore, since $B$ is a Boolean ring, we have $(n-u) \leqq (n-x)$ for every $x \in X$ and every $n \in N$. Since $y \leqq n$ for every $n \in N$ and since $B$ is a Boolean ring, it is clear that $(y-u) \leqq (n-u) \leqq (n-x)$ for every $x \in X$ and every $n \in N$. Hence $(y-u) \in D$. But then, since $u$ is an upper bound of $X \cup D$ and hence is an upper bound of $D$, we have $u \geqq (y-u)$, which implies $y \leqq u$ since $B$ is a Boolean ring. This contradicts (20). Hence our assumption is false, and $u \in N$.

Thus an element $u$ of $B$ is an upper bound of $(X \cup D)$ if and only if $u \in N$, and therefore $X \vee D = \bigcap_{v \in N} I(v) = Y$, and (18) is established. Consequently $D$ is a complement of $X$ with respect to $Y$, and the lemma is proved.

From Lemmas 4 and 5 we have

*Corollary* 1. *The conditional completion* $(L, \subseteqq)$ *by lower cuts of Boolean ring* $(B, \leqq)$ *is a sectionally complemented lattice.*

Notation. *Let $X$ and $Y$ be elements of* $(L, \subseteqq)$ *such that* $X \subseteqq Y$. *Then the complement $D$ of $X$ with respect to $Y$ as given by* (16) *is denoted by*

(21) $$Y-X.$$

We recall that a Boolean ring $(B, \leqq)$ is also a ring $(B, +, \cdot)$ in which every element is idempotent, and therefore $(B, +, \cdot)$ is commutative and of characteristic 2. In other words, for every element $x$ and $y$ of a Boolean ring $(B, +, \cdot)$, we have:

(22) $$xy = yx \quad \text{and} \quad x^2 = x \quad \text{and} \quad x+x = 0.$$

Moreover, for every element $x$ and $y$ of a Boolean ring $(B, \leqq) \equiv (B, +, \cdot)$ we have:

(23) $$x \leqq y \quad \text{if and only if} \quad xy = x$$

and

(24) $$y-x = y+yx, \quad \text{and,} \quad y-x = y+x \quad \text{if and only if} \quad x \leqq y.$$

It is worth observing that the passage from $(B, \leqq)$ to $(B, +, \cdot)$ is given by

$$x+y = (x \vee y)-(x \wedge y) \quad \text{and} \quad xy = x \wedge y.$$

Next we prove

**Lemma 6.** *For every element $X$ and $Y$ of* $(L, \subseteqq)$ *such that* $X \subseteqq Y$ *we have*

(25) $$Y-(Y-X) = X.$$

PROOF. Let $X=\bigcap_{v\in M} I(v)$ where $M$ is the set of all upper bounds of $X$, and let $Y=\bigcap_{v\in N} I(v)$ where $N$ is the set of all upper bounds of $Y$. From (16) and (21) it follows that

(26) $Y-(Y-X)=\cap\{I(v)\mid v=(n-z) \text{ for some } n\in N \text{ and for some } z\in(Y-X)\}.$

First we show that if $w\in X$ then $w\in(Y-(Y-X))$. If $w\in X$ then $w$ is a lower bound of $N$ and therefore $w\leqq n$ for every $n\in N$. On the other hand, since $B$ is a Boolean ring, then for every $z\in(Y-X)$, in view of (24), we have $w(n-z)=w-wz=w$ since, by (17) and (21) we have $X\cap(Y-X)=\{0\}$ and therefore $wz=0$. Thus by, (23) we see that $w\leqq(n-z)$ for every $n\in N$ and every $z\in(Y-X)$. Consequently, $w\in(Y-(Y-X))$, by (26).

Next, let $w$ be an element of $Y-(Y-X)$. We show that $w\in X$. Let us assume, on the contrary, that $w\notin X$. Clearly $w\in Y$ and consequently, by our assumption, $w\in(Y\setminus X)$. However, since $w\notin X$, there exists $u\in(M\setminus N)$ such that

(27) $$w\not\leqq u.$$

Since $u$ is an upper bound of $X$, we see that $x\leqq u$ for every $x\in X$ and therefore since $B$ is a Boolean ring, $(n-u)\leqq(n-x)$ for every $x\in X$ and every $n\in N$. Again, since $w\leqq n$, we have $(w-u)\leqq(n-u)\leqq(n-x)$ for every $x\in X$ and every $n\in N$. Consequently, $(w-u)\in(Y-X)$. However, since $w\in(Y-(Y-X))$ we see that $w\leqq(n-(w-n))$ for some $n\in N$. But this, in view of (23), (24) and (22) implies $wn+w+wu=w$ and since $w\leqq u$, it follows that $w=wu$. Hence, by (23) we see that $w\leqq u$ which contradicts (27). Thus, our assumption is false and indeed $w\in X$, and the lemma is proved.

**Lemma 7.** *For every element $X$ and $Y$ of $(L,\subseteqq)$ such that $X\subseteqq Y$ we have*

(28) $t\in(Y-X)$ *if and only if* $t\in Y$ *and* $tx=0$ *for every* $x\in X$

*and*

(29) $t\in X$ *if and only if* $t\in Y$ *and* $tc=0$ *for every* $c\in(Y-X)$.

PROOF. Let $t\in(Y-X)$. By (17) and (21) we have $(Y-X)\cap X=\{0\}$, and hence $tx=0$ for every $x\in X$ since $tx\in((Y-X)\cap X)$. This is because $Y-X$, as well as $X$, is a lower cut of $(L,\subseteqq)$.

Next let $t\in Y$ and $tx=0$ for every $x\in X$. Let us assume, on the contrary, that $t\notin(Y-X)$. Thus by (16) we see that $t\not\leqq(n-x)$ for some $n\in N$ and some $x\in X$ where $Y=\bigcap_{v\in N} I(v)$ and where $N$ is the set of all upper bounds of $Y$. Therefore by (23) and (24) we have $t(n+x)\neq t$ and hence $(tn+tx)\neq t$. However, since $t\in Y$ implies $t\leqq n$, we have $tn=t$ which implies $tx\neq 0$, contradicting the hypothesis that $tx=0$ for every $x\in X$. Hence our assumption is false and (28) is established.

In order to prove (29), let us observe that by (28) we have $t\in(Y-(Y-X))$ if and only if $t\in Y$ and $tc=0$ for every $c\in(Y-X)$, which in view of (25) implies (29).

**Lemma 8.** *For every element $X$, $Y$ and $Z$ of $(L,\subseteqq)$ such that $X\subseteqq Z$ and $Y\subseteqq Z$, we have*

(30) $$X\subseteqq Y \text{ if and only if } (Z-X)\supseteqq(Z-Y).$$

PROOF. Let $X \subseteqq Y$ and let $t \in (Z-Y)$. Then by (28), we have $ty=0$ for every $y \in Y$, and since $X \subseteqq Y$ we have $tx=0$ for every $x \in X$. Thus again by (28) we have $t \in (Z-X)$.

Next let $(Z-X) \supseteqq (Z-Y)$, and let $t \in X$. Then by (29) we see that $tc=0$ for every $c \in (Z-X)$. Since $(Z-X) \supseteqq (Z-Y)$ we have $td=0$ for every $d \in (Z-Y)$. But then again by (29) we have $t \in Y$, and hence $X \subseteqq Y$.

**Lemma 9.** *For every element $X$, $Y$ and $Z$ of $(L, \subseteqq)$ such that $X \subseteqq Z$ and $Y \subseteqq Z$ we have*

$$X \subseteqq Y \quad \text{if and only if} \quad X \cap (Z-Y) = \{0\}. \tag{31}$$

PROOF. Let $X \subseteqq Y$. By (30) we have $(Z-X) \supseteqq (Z-Y)$, and by (17) and (21) we see that $\{0\} = (X \cap (Z-X)) \supseteqq (X \cap (Z-Y))$, and therefore $X \cap (Z-Y) = \{0\}$.

Next let $X \cap (Z-Y) = \{0\}$, and let $x \in X$. But then $xc=0$ for every $c \in (Z-Y)$ since $xc \in (X \cap (Z-Y))$. This is because $X$ as well as $Z-Y$ is a lower cut of $(L, \subseteqq)$. Therefore by (29) we see that $x \in Y$, and hence (31) is established.

**Lemma 10.** *For every element $X$, $Y$ and $Z$ of $(L, \subseteqq)$ such that $X \subseteqq Z$ and $Y \subseteqq Z$ we have*

$$Z-(X \wedge Y) = (Z-X) \vee (Z-Y) \tag{32}$$

*and*

$$Z-(X \vee Y) = (Z-X) \wedge (Z-Y). \tag{33}$$

PROOF. Since $X \subseteqq (X \vee Y)$ and $Y \subseteqq (X \vee Y)$ it follows from (30) that $(Z-(X \vee Y)) \subseteqq (Z-X)$ and $(Z-(X \vee Y)) \subseteqq (Z-Y)$ and consequently $(Z-(X \vee Y)) \subseteqq (Z-X) \cap (Z-Y) = (Z-X) \wedge (Z-Y)$.

Thus to prove (32), it remains to show that

$$(Z-(X \vee Y)) \supseteqq (Z-X) \wedge (Z-Y). \tag{34}$$

But then in view of (31), to establish (34), it is sufficient to show that

$$((Z-X) \wedge (Z-Y)) \cap (Z-(Z-(X \vee Y))) = \{0\}$$

which in view of (10) and (25) reduces to proving that

$$(Z-X) \cap (Z-Y) \cap (X \vee Y) = \{0\}. \tag{35}$$

Let us assume the contrary, and let

$$(Z-X) \cap (Z-Y) \cap (X \vee Y) = C \neq \{0\} \tag{36}$$

for some lower cut $C$ of $(L, \subseteqq)$. We claim that

$$X \subseteqq ((X \vee Y)-C). \tag{37}$$

Let $t \in X$. Then, since $X \subseteqq Z$, by (29) we see that $tx=0$ for every $x \in (Z-X)$. Hence, $tc=0$ for every $c \in C$, since $C \subseteqq (Z-X)$, by (36). Again, by (36) we have $C \subseteqq (X \vee Y)$ and therefore, from (28) it follows that $t \in ((X \vee Y)-C)$, since $t \in X$ and $X \subseteqq (X \vee Y)$ and $tc=0$ for every $c \in C$. Thus, indeed $X \subseteqq ((X \vee Y)-C)$, and (37) is established.

Similarly, we can prove that

$$Y \subseteqq ((X\vee Y)-C). \tag{38}$$

From (37) and (38) it follows that

$$(X\vee Y) \subseteqq ((X\vee Y)-C) \subseteqq (X\vee Y)$$

which implies $C=\{0\}$ since $C\subseteqq(X\vee Y)$. This contradicts (36), and therefore (35), and hence also (32), is established.

To prove (33), we substitute $Z-X$ for $X$ and $Z-Y$ for $Y$ in (32), and in view of (25) we obtain $X\vee Y=Z-((Z-X)\wedge(Z-Y))$ which, again in view of (25), yields (33), as desired.

A dual of Lemma 9 is given by

*Corollary* 2. *For every element* $X$, $Y$ *and* $Z$ *of* $(L, \subseteqq)$ *such that* $X\subseteqq Z$ *and* $Y\subseteqq Z$ *we have*

$$X \subseteqq Y \quad \textit{if and only if} \quad ((Z-X)\vee Y) = Z. \tag{39}$$

Proof. In view of (31) we have

$$X \subseteqq Y \quad \text{if and only if} \quad Z-(X\wedge(Z-Y)) = Z-\{0\} = Z \tag{40}$$

which by (33) and (25) implies $X\subseteqq Y$ if and only if $(Z-X)\vee(Z-(Z-Y))=(Z-X)\vee \vee Y=Z$, as desired.

Based on the above lemmas, we prove the following theorem.

**Theorem 2.** *For every element* $X$, $Y$ *and* $Z$ *of* $(L, \subseteqq)$ *we have*

$$X\wedge(Y\vee Z) = (X\wedge Y)\vee(X\wedge Z) \tag{41}$$

*and*

$$X\vee(Y\wedge Z) = (X\vee Y)\wedge(X\vee Z). \tag{42}$$

Proof. Since $X\wedge(Y\vee Z)\supseteqq(X\wedge Y)$ and $X\wedge(Y\vee Z)\supseteqq(X\wedge Z)$ it is clear that $X\wedge(Y\vee Z)\supseteqq((X\wedge Y)\vee(X\wedge Z))$. Hence to establish (41) it remains to show that

$$X\wedge(Y\vee Z) \subseteqq ((X\wedge Y)\vee(X\wedge Z)). \tag{43}$$

Let $H=X\vee Y\vee Z$. Then

$$\begin{aligned} &X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z)))\wedge Y = \\ &= X\wedge(Y\vee Z)\wedge(H-(X\wedge Y))\wedge(H-((X\wedge Z)))\wedge Y = \quad \text{by (33)} \\ &= ((X\wedge Y)\wedge(H-(X\wedge Y)))\wedge(Y\vee Z)\wedge(H-(X\wedge Z)) = \{0\} \quad \text{by (17) and (21).} \end{aligned} \tag{44}$$

Therefore, by (25) we have:

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z)))\wedge(H-(H-Y)) = \{0\}$$

which by (31 implies:

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq (H-Y). \tag{45}$$

Similarly, by replacing the last $Y$ appearing in (44) by $Z$, we obtain

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z)))\wedge Z = \{0\}$$

which, as in the case of (45), implies

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq (H-Z). \tag{46}$$

But then, from (45) and (46) it follows that

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq ((H-Y)\wedge(H-Z))$$

which by (33) implies

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq (H-(Y\vee Z)). \tag{47}$$

On the other hand, it is clear that

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq (Y\vee Z)$$

which, in view of (47) and (17) implies

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) \subseteqq ((H-(Y\vee Z))\wedge(Y\vee Z)) = \{0\}$$

and consequently,

$$X\wedge(Y\vee Z)\wedge(H-((X\wedge Y)\vee(X\wedge Z))) = \{0\}$$

which by (31) implies (43). But then (41) follows, as mentioned above.

To prove (42), we replace $X$, $Y$ and $Z$ respectively by $H-X$, $H-Y$ and $H-Z$ in (41). As a result, we obtain

$$(H-X)\wedge((H-Y)\vee(H-Z)) = ((H-X)\wedge(H-Y))\vee((H-X)\wedge(H-Z))$$

which by (32) and (33) implies

$$(H-X)\wedge(H-(Y\wedge Z)) = (H-(X\vee Y))\vee(H-(X\vee Z))$$

which, in turn, by (25), (32) and (33) implies

$$X\vee(Y\wedge Z) = H-((H-X)\wedge(H-(Y\wedge Z))) = H-((H-(X\vee Y))\vee(H-(X\vee Z))) =$$
$$=(X\vee Y)\wedge(X\vee Z)$$

as desired. Thus, (42) is established.

The above Theorem establishes the distributivity of $(L, \subseteqq)$. Observing that in a distributive lattice (sectional) complementation is unique, based of Lemmas 3, 4, 5 and Theorems 1 and 2, we have:

**Theorem 3.** *The conditional completion $(L, \subseteqq)$ by lower cuts of a Boolean ring $(B, \leqq)$ is a sectionally complemented distributive lattice and hence a conditionally complete Boolean ring. Moreover, the mapping $f$ from $B$ into $L$ defined by*

$$f(x) = I(x) = \{y \mid y\in B \text{ and } y \leqq x\}$$

*is a ring isomorphism from $(B, \leqq)$ into $(L, \subseteqq)$ such that $f$ preserves suprema (and hence, also infima).*

*Remark* 2. Let us observe that if the Boolean ring $(B, \leqq)$ in Theorem 3 is a Boolean algebra (i.e., $B$ has a multiplicative unit 1), then the conditional completion ($L$, $\subseteqq$ by) lower cuts of $(B, \leqq)$ is in fact a complete Boolean algebra (see Remark 1 on page 255). Thus, to obtain a completion of a Boolean ring $(B, \leqq)$ which is not a Boolean algebra, it is sufficient to consider the conditional completion by lower cuts of the Boolean algebra $(A, \leqq)$ which is obtained by adjoining to $B$ a new symbol 1 (as a multiplicative unit) along with $x+1$ for every $x \in B$ and by defining addition and multiplication in $A$ in an obvious way.

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