Generalisation of Gauss-Codazzi equations for Berwald's curvature tensor in a hypersurface of a Finsler space

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Introduction

In the present paper the authors wish to derive the generalisations of Gauss Codazzi equations by considering the induced Berwald's covariant derivative of a unit vector in the direction of the congruences of curves associated to a hypersurface of a Finsler space.

1. Notations and fundamental formulae

Let F_n be an n-dimensional Finsler space equipped with a positively homogeneous metric function $F(x, \dot{x})$. The fundamental metric tensor $g_{ij}(x, \dot{x})$ of F_n is defined by [1][1])

(1.1)
$$g_{ij}(x, \dot{x}) = \frac{1}{2}\dot{\partial}^2_{ij} F^2(x, \dot{x}),\ ^{2})$$

where

(1.2)
$$C_{ijk}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2}\dot{\partial}_k g_{ij}(x, \dot{x}).$$

Let $X^i(x, \dot{x})$ be a vector field. The covariant derivatives of $X^i(x, \dot{x})$ with respect to x^k in the sense of Bernwald and Cartan are given by

(1.4)
$$X^i_{(k)} = \partial_k X^i - (\dot{\partial}_m X^i) G^m_{k\gamma}\dot{x}^\gamma + X^m G^i_{mk}$$

and

(1.5)
$$X^i_{|k} = \partial_k X^i - (\dot{\partial}_m X^i) \Gamma^{*m}_{\gamma k}\dot{x}^\gamma + X^m \Gamma^{*i}_{mk}$$

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[1]) Numbers in brackets refer to the references given at the end of the paper.

[2]) $\dot{\partial}_i = \partial/\partial\dot{x}^i$, $\partial_i = \partial/\partial x^i$ and $\dot{\partial}^2_{ij} = \partial^2/\partial\dot{x}^i\partial\dot{x}^j$.

respectively, where $G^i_{mk}(x, \dot{x})$ and $\Gamma^{*i}_{mk}(x, \dot{x})$ are connection parameters symmetric in its lower indices and are related by the equation

$$\Gamma^{*i}_{\gamma k}\dot{x}^{\gamma} = G^i_{\gamma k}\dot{x}^{\gamma}. \tag{1.6}$$

We shall now consider a hypersurface F_{n-1} of F_n represented by

$$x^i = x^i(u^{\alpha}), \quad (\alpha = 1, 2, \ldots, n-1) \tag{1.7}$$

where u^{α} are the parameter. The matrix $\|B^i_{\alpha}\|$ of the projection parameters has rank $n-1$. We shall use the following notations

$$B^i_{\alpha} = \partial_{\alpha} x^i, \quad B^i_{\alpha\beta} = \partial^2_{\alpha\beta} x^i, \quad B^{ij\ldots k}_{\alpha\beta\ldots\gamma} = B^i_{\alpha} B^j_{\beta} \ldots B^k_{\gamma}. \tag{1.8}$$

The hypersurface vector $\dot{u}^{\alpha}$ and a vector $\dot{x}^i$ of F_n are related by

$$\dot{x}^i = \dot{u}^{\alpha} B^i_{\alpha}. \tag{1.9}$$

The metric (induced) tensor $g_{\alpha\beta}(u, \dot{u})$ of F_{n-1} is given by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x}) B^{ij}_{\alpha\beta}. \tag{1.10}$$

$g^{\alpha\beta}(u, \dot{u})$ are the inverse of the tensor $g_{\alpha\beta}(u, \dot{u})$; therefore we can construct the following entities:

$$B^{\alpha}_i = g^{\alpha\epsilon} B^j_{\epsilon} g_{ij} \tag{1.11}$$

which satisfies the identity

$$B^{\alpha}_i B^i_{\beta} = \delta^{\alpha}_{\beta} = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta. \end{cases} \tag{1.12}$$

At each point P of F_{n-1} we can define the unit normal vectors $N^i(x, \dot{x})$ with respect to the tangential direction $\dot{x}^i$ at P by the set of equations.

$$\text{a)}\ g_{ij} N^j B^i_{\alpha} = N_i B^i_{\alpha} = 0 \qquad \text{b)}\ g^{ij} N_i N_j = N^i N_i = 1 \tag{1.13}$$

which imply

$$\text{a)}\ N^i B^{\alpha}_i = 0, \qquad \text{b)}\ g_{ij} N^i N^j = 1. \tag{1.14}$$

The induced symmetric tensor $A^{\alpha}_{\beta\gamma}$ and the connection parameters $G^{\alpha}_{\beta\gamma}$ (Sinha and Singh [3]) of F_{n-1} are given by

$$A^{\alpha}_{\beta\gamma}(u, \dot{u}) = A^i_{hk}(x, \dot{x}) B^{\alpha}_i B^{hk}_{\beta\gamma} \tag{1.15}$$

and

$$G^{\alpha}_{\beta\gamma} = B^{\alpha}_i \{B^i_{\beta\gamma} + B^{hk}_{\beta\gamma}(G^i_{hk} - C^i_{hk|m}\dot{x}^m)\} + \Lambda^{\alpha}_{\beta\gamma} + C^{\alpha}_{\beta\gamma|\varrho}\dot{u}^{\varrho}, \tag{1.16}$$

where the quantity $\Lambda^{\alpha}_{\beta\gamma} = g^{\alpha\varepsilon}\Lambda_{\beta\varepsilon\gamma}$ satisfies the identities

$$\Lambda^{\alpha}_{\beta\gamma}\dot{u}^{\beta} = NM^{\alpha}_{\gamma} \quad \text{and} \quad \Lambda^{\varepsilon}_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta} = 0.$$

With the help of the above quantities we can define an induced mixed derivative denoted by $T^i_{\alpha((\beta))}$ in the sense of Berwald as follows

$$T^i_{\alpha((\beta))} = \partial_{\beta} T^i_{\alpha} - (\dot{\partial}_{\gamma} T^i_{\alpha}) G^{\gamma}_{\beta} + T^s_{\alpha} G^i_{sh} B^h_{\beta} - T^i_{\varepsilon} G^{\varepsilon}_{\alpha\beta}. \tag{1.17}$$

With the help of the above equation, we can define the mixed tensor $V^i_{\alpha\beta}$ as

$$B^i_{\alpha((\beta))} \overset{\text{def}}{=} V^i_{\alpha\beta} = B^i_{\alpha\beta} - B^i_\varepsilon G^\varepsilon_{\alpha\beta} + G^i_{hk} B^{hk}_{\alpha\beta} \tag{1.18}$$

which are regarded as vectors of the imbeding space F_{n-1} and are normal to F_{n-1}. We may write $V^i_{\alpha\beta}$, [3], as

$$V^i_{\alpha\beta} = N^i \tilde{\Omega}_{\alpha\beta} - B^i_\varepsilon T^\varepsilon_{\alpha\beta} + C^i_{hk|m} \dot{x}^m B^{hk}_{\alpha\beta}, \tag{1.19}$$

where $\tilde{\Omega}_{\alpha\beta}$ is the second fundamental tensor symmetric in its lower indices and

$$T^\varepsilon_{\alpha\beta} \overset{\text{def}}{=} \Lambda^\varepsilon_{\alpha\beta} + C^\varepsilon_{\alpha\beta|\varrho} \dot{u}^\varrho. \tag{1.20}$$

The induced derivative $N^i_{((\beta))}$ of type (1.17) is given by (Sinha and Singh [3]):

$$N^i_{((\beta))} = -\tilde{\Omega}_{\alpha\beta} g^{\alpha\delta} B^i_\delta + E^i_m V^m_{\varrho\beta} \dot{u}^\varrho + N^h C^i_{hk|m} \dot{x}^m B^k_\beta, \tag{1.21}$$

where

$$E^i_m = M_m N^i - 2M^i_m \tag{1.22}$$

and

$$M^i_l = C^i_{lp} N^p, \quad M_l = C_{pkl} N^p N^k = M_{kl} N^k. \tag{1.23}$$

2. Generalised Gauss Codazzi equations

Consider a set of congruences of curves such that one curve of each of them passes throught every point of F_{n-1}. We consider the contravariant components of a unit vector in the direction of a curve of a congruence of curves as a linear combination of tangent vector B^i_α and normal vector N^i as

$$\lambda^i = t^\alpha B^i_\alpha + dN^i, \tag{2.1}$$

where t^α and d are parameters. Taking the mixed derivative of λ^i with respect to u^β of type (1.17) we get

$$\lambda^i_{((\beta))} = B^i_\alpha t^\alpha_{((\beta))} + t^\alpha B^i_{\alpha((\beta))} + N^i d_{((\beta))} + dN^i_{((\beta))}, \tag{2.2}$$

again taking the covariant derivative of (2.1) with respect to u^γ of type (1.17) and substracting the equation obtained by interchanging the indices β and γ we get

$$\lambda^i_{[((\beta))((\gamma))]} = B^i_\alpha t^\alpha_{[((\beta))((\gamma))]} + t^\alpha B^i_{\alpha[((\beta))((\gamma))]} + dN^i_{[((\beta))((\gamma))]} + N^i d_{[((\beta))((\gamma))]}. \tag{2.3}$$

With the help of equation (1.18) we can write

$$B^i_{\alpha[((\beta))((\gamma))]} = V^i_{\alpha[\beta((\gamma))]}. \text{ [3]} \tag{2.4}$$

Thus by substituting equations (1.19) and (1.21) in (2.4) we obtain

$$\begin{aligned} B^i_{\alpha[((\beta))((\gamma))]} = {} & B^i_\delta \{T^\varepsilon_{\alpha[\beta} T^\delta_{\langle\varepsilon\rangle\gamma]} - \tilde{\Omega}_{\alpha[\beta} \tilde{\Omega}_{\gamma]\varepsilon} g^{\varepsilon\delta} - T^\delta_{\alpha[\beta((\gamma))]}\} + \\ & + N^i \{\tilde{\Omega}_{\alpha[\beta((\gamma))]} - T^\delta_{\alpha[\beta} \Omega_{\gamma]\delta}\} + C^i_{hk|m} \dot{x}^m \{\tilde{\Omega}_{\alpha[\beta} B^k_{\gamma]} N^h - T^\delta_{\alpha[\beta} B^{hk}_{\gamma]\delta}\} + \\ & + E^i_m V^m_{\varrho[\gamma} \tilde{\Omega}_{\beta]\alpha} \dot{u}^\varrho + (C^i_{hk|m} \dot{x}^m B^{hk}_{\alpha[\beta})_{((\gamma))]}. \text{ [4]} \end{aligned} \tag{2.5}$$

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[3]) $2x_{[\alpha\beta]} = x_{\alpha\beta} - x_{\beta\alpha}$.

[4]) Indices in brackets $\langle\ \rangle$ are free from symmetric and skew symmetric parts.

Similarly, by using equation (1.19) and (1.21) we get

$$\begin{aligned} N^i_{[((\beta))((\gamma))]} &= B^i_\delta\{T^\delta_{\varepsilon[\gamma}\tilde{\Omega}_{\beta]\alpha}g^{\alpha\varepsilon} - (g^{\alpha\delta}\tilde{\Omega}_{\alpha[\beta})_{((\gamma))]}\} + \\ &+ E^i_m(V^m_{\varrho[\beta}\dot{u}^\varrho)_{((\gamma))]} + E^i_{m[((\gamma))}V^m_{\beta]\varrho}\dot{u}^\varrho + (C^i_{hk|m}\dot{x}^m)_{[((\gamma))}B^k_{\beta]}N^h + \\ &+ C^i_{hk|m}\dot{x}^m\{(E^h_s V^s_{\varrho[\gamma}\dot{u}^\varrho + N^l C^h_{l\,s|a}\dot{x}^a B^s_{[\gamma})B^k_{\beta]}\}. \end{aligned} \tag{2.6}$$

We have $\lambda^i_{((\beta))} = \lambda^i_{(h)}B^h_\beta$ and $\lambda^i_{((\beta))((\gamma))} = \lambda^i_{(h)(k)}B^{hk}_{\beta\gamma} + \lambda^i_{(h)}V^h_{\beta\gamma}$.

Thus we have

$$\lambda^i_{[((\beta))((\gamma))]} = \lambda^i_{[(h)(k)]}B^{hk}_{\beta\gamma}. \tag{2.7}$$

But we know the following commutation formula [1]

$$2\lambda^i_{[(h)(k)]} = \lambda^j H^i_{jhk} - (\dot{\partial}_j\lambda^i)H^j_{hk}. \tag{2.8}$$

From equations (2.7) and (2.8), we get

$$2\lambda^i_{[((\beta))((\gamma))]} = (\lambda^j H^i_{jhk} - \dot{\partial}_j\lambda^i H^j_{hk})B^{hk}_{\beta\gamma}. \tag{2.9}$$

Similarly for t^δ we have

$$2t^\delta_{[((\beta))((\gamma))]} = t^\varepsilon H^\delta_{\varepsilon\beta\gamma} - (\dot{\partial}_\varepsilon t^\delta)H^\varepsilon_{\beta\gamma}, \tag{2.10}$$

where

$$H^\delta_{\varepsilon\beta\gamma}(u, \dot{u}) = 2\{\partial_{[\gamma}G^\delta_{\beta]\varepsilon} + G^\varrho_{\varepsilon[\beta}G^\delta_{\gamma]\varrho} + G^\delta_{\varrho\varepsilon[\gamma}G^\varrho_{\beta]}\} \tag{2.11}$$

and

$$H^\varepsilon_{\beta\gamma}(u, \dot{u}) = 2\{\partial_{[\gamma}\dot{\partial}_{\beta]}G^i + G^\varepsilon_{\varrho[\gamma}G^\varrho_{\beta]}\}. \tag{2.12}$$

The induced covariant derivative of d in the sense of Berwald is given by

$$d_{((\beta))} = \partial_\beta d - (\dot{\partial}_\gamma d)G^\gamma_\beta. \tag{2.13}$$

Therefore, we get

$$d_{[((\beta))((\gamma))]} = \dot{\partial}_\varrho d\{\partial_{[\beta}G^\varrho_{\gamma]} + G^\varrho_{\varepsilon[\beta}G^\varepsilon_{\gamma]}\}. \tag{2.14}$$

Substituting equations (2.5), (2.6), (2.9), (2.10) and (2.14) in (2.3) we obtain

$$\begin{aligned} \lambda^j H^i_{jhk}B^{hk}_{\beta\gamma} &= H^\delta_{\varepsilon\beta\gamma}t^\varepsilon B^i_\delta + (\dot{\partial}_n\lambda^i)H^n_{hk}B^{hk}_{\beta\gamma} - B^i_\delta(\dot{\partial}_\varepsilon t^\delta)H^\varepsilon_{\beta\gamma} + \\ &+ 2\big[B^i_\delta\big\{t^\alpha(T^\varepsilon_{\alpha[\beta}T^\delta_{\langle\varepsilon\rangle\gamma]} - \tilde{\Omega}_{\alpha[\beta}\tilde{\Omega}_{\gamma]\varepsilon}g^{\varepsilon\delta} - T^\delta_{\alpha[\beta((\gamma))]}) + d\big(T^\delta_{\varepsilon[\gamma}\tilde{\Omega}_{\beta]\alpha}g^{\alpha\varepsilon} + (g^{\alpha\delta}\tilde{\Omega}_{\alpha[\beta})_{((\gamma))]}\big)\big\} + \\ &+ N^i\{t^\alpha(\tilde{\Omega}_{\alpha[\beta((\gamma))]} - T^\delta_{\alpha[\beta}\tilde{\Omega}_{\gamma]\delta}) + \dot{\partial}_\varrho d(\partial_{[\beta}G^\varrho_{\gamma]} + G^\varrho_{\varepsilon[\beta}G^\varepsilon_{\gamma]})\} + \\ &+ E^i_m\{t^\alpha V^m_{\varrho[\gamma}\tilde{\Omega}_{\beta]\alpha}\dot{u}^\varrho + d(V^m_{\varrho[\beta}\dot{u}^\varrho)_{((\gamma))]}\} + C^i_{hk|m}\dot{x}^m\{t^\alpha(\tilde{\Omega}_{\alpha[\beta}B^k_{\gamma]}N^h - T^\delta_{\alpha[\beta}B^{hk}_{\gamma]\delta}) + \\ &+ d(E^h_S V^S_{\varrho[\gamma}\dot{u}^\varrho + N^\varrho C^h_{\varrho S|a}\dot{x}^a B^S_{[\gamma})B^k_{\beta]}\} + t^\alpha(C^i_{hk|m}\dot{x}^m B^{hk}_{\alpha[\beta})_{((\gamma))]} + \\ &+ d\{E^i_{m[((\gamma))}V^m_{\beta]\varrho}\dot{u}^\varrho + (C^i_{hk|m}\dot{x}^m)_{((\gamma))}B^k_{\beta]}N^h\}\big]. \end{aligned} \tag{2.15}$$

Multiplying equation (2.15) by B_i^δ and using equations (1.12), (1.14) and (1.22) we obtain

$$
\begin{aligned}
\lambda^n H^i_{nhk} B^{hk}_{\beta\gamma} B^\delta_i &= H^\delta_{\varepsilon\beta\gamma} t^\varepsilon + B^\delta_i (\dot\partial_n \lambda^i) H^n_{hk} B^{hk}_{\beta\gamma} - (\dot\partial_\varepsilon t^\delta) H^\varepsilon_{\beta\gamma} + \\
&+ 2\big[t^\alpha \big\{(T^\varepsilon_{\alpha[\beta} T^\delta_{\langle\varepsilon\rangle\gamma]} - \tilde\Omega_{\alpha[\beta} \bar\Omega_{\gamma]\varepsilon} g^{\varepsilon\delta} - T^\delta_{\alpha[\beta((\gamma))]}) + d\big(T^\delta_{\varepsilon[\gamma} \tilde\Omega_{\beta]\alpha} g^{\alpha\delta} + (g^{\alpha\delta} \tilde\Omega_{\alpha[\beta})_{((\gamma))]}\big)\big\} + \\
&+ B^\delta_i \big\{C^i_{hk|m} \dot x^m \big(t^\alpha (\tilde\Omega_{\alpha[\beta} B^k_{\gamma]} N^h - T^\delta_{\alpha[\beta} B^{hk}_{\gamma]\delta}) + d(E^h_S V^S_{\varrho[\gamma} \dot u^\varrho + N^j C^h_{jS|a} \dot x^a B^S_{[\gamma)}) B^k_{\beta]}\big) - \\
&- 2M^i_m \big(t^\alpha V^m_{\varrho[\gamma} \tilde\Omega_{\beta]\alpha} \dot u^\varrho + d(V^m_{\varrho[\beta} \dot u^\varrho)_{((\gamma))]}\big) + t^\alpha (C^i_{hk|m} \dot x^m B^{hk}_{\alpha[\beta)})_{((\gamma))} + \\
&+ d\big(E^i_{m[((\gamma))} V^m_{\beta]\varrho} \dot u^\varrho + (C^i_{hk|m} \dot x^m)_{[((\gamma))} B^k_{\beta]} N^h\big)\big\}\big].
\end{aligned} \tag{2.16}
$$

Again multiplying (2.15) by N_i and using equation (1.13) we get

$$
\begin{aligned}
N_i \lambda^n H^i_{nhk} B^{hk}_{\beta\gamma} &= (\dot\partial_n \lambda^i) N_i H^n_{hk} B^{hk}_{\beta\gamma} + 2\big[t^\alpha (\tilde\Omega_{\alpha[\beta((\gamma))]} - T^\delta_{\alpha[\beta} \bar\Omega_{\gamma]\delta}) + \\
&+ \dot\partial_\varrho d\{\partial_{[\beta} G^\varrho_{\gamma]} + G^\varrho_{\varepsilon[\beta} G^\varepsilon_{\gamma]}\} + N_i \big\{E^i_m \big(t^\alpha V^m_{\varrho[\gamma} \tilde\Omega_{\beta]\alpha} \dot u^\varrho + d(V^m_{\varrho[\beta} \dot u^\varrho)_{((\gamma))]}\big) + \\
&+ C^i_{hk|m} \dot x^m \big(t^\alpha (\tilde\Omega_{\alpha[\beta} B^k_{\gamma]} N^h - T^\delta_{\alpha[\beta} B^{hk}_{\gamma]\delta}) + d(E^h_S V^S_{\varrho[\gamma} \dot u^\varrho + N^j C^h_{jS|a} \dot x^a B^S_{[\gamma)}) B^k_{\beta]}\big) + \\
&+ t^\alpha (C^i_{hk|m} \dot x^m B^{hk}_{\alpha[\beta)})_{((\gamma))]} + d\big(E^i_{m[((\gamma))} V^m_{\beta]\varrho} \dot u^\varrho + (C^i_{hk|m} \dot x^m)_{[((\gamma))} B^k_{\beta]} N^h\big\}\big].
\end{aligned} \tag{2.17}
$$

Equations (2.16) and (2.17) can be regarded as generalisation of Gauss Codazzi equation in a hypersurface F_{n-1} imbedded in a Finsler space F_n.

3. Particular cases

We can choose a congruence of curves in three different ways. Firstly it is supposed to be normal to F_{n-1} i.e. $\lambda^i = dN^i$. Secondly it is such that the vector with the contravariant components λ^i in the direction of the curve of the congruence is only depending upon the tangent vectors B^i_α i.e. $\lambda^i = t^\alpha B^i_\alpha$; thirdly it can be tangential to the curve $\dot x^i = \dot u^\alpha B^i_\alpha$.

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