A new algebra of distributions; initial-value problems involving Schwartz distributions. I

By GREGERS KRABBE (West Lafayette, Ind.)

The theme of this paper is a space $\mathfrak{B}$ of distributions; this space $\mathfrak{B}$ is closed under convolution (the definition of this particular convolution reguires no restriction on the supports — nor does it require growth conditions). The space $\mathfrak{B}$ contains all the functions which are locally integrable on $(-\infty, \infty)$; the space $\mathfrak{B}$ also contains D'_+ and each derivative of every distribution whose support is a locally finite subset of $(-\infty, \infty)$. If T is a distribution whose derivative ∂T belongs to $\mathfrak{B}$, then T also belongs to $\mathfrak{B}$, and T has a well-defined *initial-value* $T(0-)$. Thus, it is possible to consider initial-value problems involving arbitrary distributions whenever the input belongs to the space $\mathfrak{B}$. Also defined in this paper is a one-to-one transformation of $\mathfrak{B}$ into a commutative algebra of operators (this transformation is somewhat analogous to the Fourier transformation); this gives an operational calculus which yields an existence and uniqueness theorem for differential equations subject to initial conditions of the form

$$u(0-) = c_0, \quad \partial u(0-) = c_1, \quad \partial^2 u(0-) = c_2, \dots,$$

where $c_0, c_1, c_2, \dots$ are arbitrary constants and where u is an arbitrary distribution satisfying a differential equation whose right-hand side belongs to the space $\mathfrak{B}$. The operational calculus is applied to a differential equation which cannot be solved explicitly by means of the Fourier transformation.

The new algebra is denoted $\mathfrak{B}$: it is a commutative algebra (of distributions) under convolution multiplication; the space $\mathfrak{B}$ contains D'_+ and all locally integrable functions. The space $\mathfrak{B}$ is closed under convolution and contains each derivative of every distribution whose support is locally finite. If F is a distribution whose derivative ∂F belongs to $\mathfrak{B}$, then F also belongs to $\mathfrak{B}$ and F equals a continuous function f in some interval $(a, 0)$ (with $a<0$): in consequence, F has a well-defined initial value $F(0-)=f(0-)$.

Given arbitrary constants $(a_0, a_1, \dots, a_m)$ and an arbitrary element S of the algebra $\mathfrak{B}$, we shall describe a calculus to obtain explicit solutions of differential equations of the form

$$a_m \partial^m u + \dots + a_1 \partial u + a_0 u = S \tag{1}$$

subject to initial conditions such as

$$u(0-) = c_0, \quad \partial u(0-) = c_1, \dots, \partial^{m-1} u(0-) = c_{m-1}, \tag{2}$$

where $c_0, c_1, \ldots, c_{m-1}$ are arbitrary constants. As we shall see, such initial-value problems can be solved by means of an operational calculus which is a useful substitute for the two-sided distributional Laplace transformation: it requires *no growth restrictions* and *no restrictions on the supports*. We shall prove that the equation (1) implies that both u and $\partial^k u$ belong to $\mathfrak{B}$ for each integer $k \leqq m$; moreover, there exists a unique distribution u satisfying the problem (1)—(2): this distribution belongs to the algebra $\mathfrak{B}$.

The *algebra* $\mathfrak{B}$ contains as a subalgebra the space D'_+ (of all the distributions which vanish on the interval $(-\infty, 0)$). In fact, $\mathfrak{B} = D'_+ + \mathfrak{B}_-$, where $\mathfrak{B}_-$ is the space of all the elements of D'_- which are regular in some neighborhood of the origin (here D'_- denotes the space of all the distributions which vanish on $(0, \infty)$). The space $\mathfrak{B}$ contains the space L^{loc} of all the complex-valued functions which are locally integrable on $(-\infty, \infty)$:

$$L^{\text{loc}} \subset \mathfrak{B} \subset D'.$$

The space $\mathfrak{B}$ also contains all the "opérateurs de Heaviside" [2]. To each F in $\mathfrak{B}$ there corresponds a unique pair (F_+, F_-) in the cartesian product $D'_+ \times \mathfrak{B}_-$ such that $F = F_+ + F_-$.

The algebra $\mathfrak{B}$ is the result of providing the space $\mathfrak{B}$ with the multiplication

$$\mathfrak{B} \times \mathfrak{B} \ni (F, G) \to F \otimes G \in \mathfrak{B}$$

defined by

$$F \otimes G = F_+ * G_+ - F_- * G_-,$$

where $*$ is convolution in the usual sense [4, p. 348]. Thus, $\otimes$ is the multiplication of the algebra $\mathfrak{B}$; it is commutative and commutes with the distributional differentiator ∂; note the absence of any restriction on the supports (there are no growth conditions either). A particular solution of the differential equation (1) is given by the Duhamel-type formula

$$u = G \otimes S,$$

where G is the regular distribution corresponding to the Green's function of the equation (1): see 5.15.

If $[f]^0$ (respectively, $[g]^0$) is the regular distribution corresponding to the locally integrable function $f(\,)$ (respectively, $g(\,)$), then $[f]^0 \otimes [g]^0$ is the regular distribution $[f \wedge g]^0$ corresponding to the function $f \wedge g(\,)$ defined by

$$f \wedge g(t) = -\int_t^0 f(t-\tau) g(\tau)\, d\tau \quad (\text{for } -\infty < t < \infty).$$

Our *operational calculus* (see 3.8) is an algebraic isomorphism of $\mathfrak{B}$ into the operator-algebra $\mathscr{A}_{\mathbf{R}}$ (see [6]); it applies to problems such as (1)—(2); for example, it yields the solution

$$u = c_0 \cos t + (c_1 + 1) \sin t + \left[\frac{t}{2\pi}\right] \sin t \tag{3}$$

of the initial-value problem

$$\partial^2 u + u = \sum_{k=-\infty}^{\infty} \delta_{2k\pi} \quad \text{subject to} \quad u(0-) = c_0 \tag{4}$$

and $\partial u(0-)=c_1$ (with c_0 and c_1 arbitrary complex numbers): as usual, $[t/2\pi]$ is the greatest integer $<t/2\pi$, and $\delta_{2k\pi}$ is the Dirac distribution concentrated at the point $2k\pi$. The first equation in (4) is a counter-example in [1, p. 128] and cannot be solved by the method of fundamental (or "elementary") solutions; its solution (3) cannot be obtained by using the Fourier transformation, the finite Fourier transformation, nor the two-sided distributional Laplace transformation.

Organization of this paper. The first sections are devoted to the operational calculus; § 5 deals with initial-value problems and contains the main results; the above example is discussed in § 6.

Concluding remarks. The theorems in § 5 resemble (and were inspired by) the ones in César de Freitas' article [2] (which deals with a space $\mathfrak{M}$ containing L^{loc} and properly contained in $\mathfrak{B}$; each element of $\mathfrak{M}$ is a linear combination of a function with a sum of distributions of finite order whose supports are locally finite). Harris Shultz gave me the idea that $F\otimes G$ belongs to $\mathfrak{B}$ whenever F and G belong to $\mathfrak{B}$; he also gave me a more elegant characterization of the space $\mathfrak{B}$ (which I use as a definition).

§ 1. Preliminaries

Let D be the space $D(\mathbf{R})$ of Schwartz test-functions on $\mathbf{R}=(-\infty, \infty)$; as usual, D' is the dual of D (see [4, p. 313]); a *distribution* is an element of D'.

If J is a subset of $\mathbf{R}$, the relation $\varphi<J$ will mean that $\varphi(\,)\in D$ and the support of $\varphi(\,)$ is a compact subset of J. We always have $\varphi<\mathbf{R}$.

1.1. *The symbol* $\bigcirc$. If F is a distribution, there exists a largest open set $\bigcirc(F)$ such that $F(\varphi)=0$ whenever $\varphi<\bigcirc(F)$ (see [4, p. 318]).

1.2. If (α, β) is an open interval, then $\varphi<(\alpha, \beta)$ if (and only if) $\varphi(\,)$ vanishes outside some closed sub-interval of (α, β).

1.3. Again, let F be a distribution. If J_1 and J_2 are open subsets of $\mathbf{R}$ such that $\bigcirc(F)\supset J_k$ for $k=1$ and $k=2$, then

$$\bigcirc(F) \supset J_1\cup J_2 \qquad\qquad \text{(see [8, pp. 27—28]).}$$

1.4. *Equality of distributions in a set.* If $J\subset\mathbf{R}$, a distribution F is said to *equal* F_1 *in* J if (and only if) $F(\varphi)=F_1(\varphi)$ whenever $\varphi<J$.

Thus, $\bigcirc(F)$ is the largest open set J such that F equals $\mathbf{0}$ in J. Note that $\mathbf{0}$ is the zero distribution.

1.5. *Note.* Let J be an open subset of $\mathbf{R}$; clearly,

$$\bigcirc(F)\supset J \quad \text{if (and only if) } F \text{ equals } \mathbf{0} \text{ in } J.$$

1.6. Lemma. *Let c_1 and c_2 be complex numbers. If F_1 and F_2 are distributions, then*

$$\bigcirc(c_1F_1+c_2F_2)\supset \bigcirc(F_1)\cap \bigcirc(F_2):$$

see [4, p. 318, Proposition 2].

1.7. *Two spaces of regular distributions.* Let $\mathscr{F}$ be the space L^{loc} of all the complex-valued functions which are locally integrable on $\mathbf{R}$. There is a linear injection $f() \to [f]^0$ of $\mathscr{F}$ into the space D' of distributions, the distribution $[f]^0$ being defined by

$$[f]^0(\varphi) = \int_{-\infty}^{\infty} f(u)\varphi(u)\,du \quad (\text{for } \varphi() \text{ in } D):$$

see p. 48 in [9], where the distribution $[f]^0$ is denoted T_f. If $\mathscr{G} \subset \mathscr{F}$ we set

$$[\mathscr{G}]^0 = \{[f]^0 : f() \in \mathscr{G}\}.$$

The elements of the space $[\mathscr{F}]^0$ are usually called *regular distributions*. Let $\mathscr{F}_-$ be the space of all the functions $f()$ in $\mathscr{F}$ such that $f()$ equals zero almost-everywhere on $(0, \infty)$ (this will henceforth be written: $f()=0$ on $(0, \infty)$):

$$\mathscr{F}_- = \{f() \in \mathscr{F} : f() = 0 \text{ on } (0, \infty)\}. \tag{1.8}$$

Thus, $B \in [\mathscr{F}_-]^0$ if B is a regular distribution $[f]^0$ such that $f()=0$ on $(0, \infty)$:

$$[\mathscr{F}_-]^0 = \{[f]^0 : f() \in \mathscr{F} \text{ and } f() = 0 \text{ on } (0, \infty)\}. \tag{1.9}$$

1.10. Let Ω be an open subset of $\mathbf{R}$ and let $f_1()$ and $f_2()$ be two elements of $\mathscr{F}$. *If* $f_1()=f_2()$ *on* Ω, *then the distribution* $[f_1]^0$ *equals* $[f_2]^0$ *in* Ω. See [9, p. 48].

1.11. Conversely, *if* $[f_1]^0$ *equals* $[f_2]^0$ *in* Ω, *then* $f_1()=f_2()$ *on* Ω (that is, the functions are equal almost-everywhere on Ω). See [9, p. 48].

1.12. Definition. *Let* $\mathfrak{B}_-$ *be the family of all the distributions which equal* $\mathbf{0}$ *in the interval* $(0, \infty)$ *and which equal some regular distribution in some interval* $(a, -a)$ *with* $a<0$.

1.13. Definitions. *Let* $\mathfrak{B}_+$ *be the family of all the distributions which equal* $\mathbf{0}$ *in the interval* $(-\infty, 0)$. *We set*

$$\mathfrak{B} = \mathfrak{B}_- + \mathfrak{B}_+; \tag{1.14}$$

thus, the family $\mathfrak{B}$ consists of all the sums $F+R$ such that $F \in \mathfrak{B}_-$ and $R \in \mathfrak{B}_+$.

1.15. Definition. *A "left-distribution" is a distribution which equals zero in some interval* (a, ∞) *containing the point* 0. We denote by $(\mathscr{L})$ the family of left-distributions.

1.16. *Therefore*, $L \in (\mathscr{L})$ if (and only if) L is a distribution which equals $\mathbf{0}$ in some interval (a, ∞) with $a<0$. In other words, $L \in (\mathscr{L})$ if (and only if) $L \in D'$ and there exists a number $a<0$ such that $\bigcirc(L) \supset (a, \infty)$.

1.17. *Reorientation.* The main object of this section is to prove that 1.14 is a direct sum. We shall also need the following lemma, which characterizes $\mathfrak{B}_-$ as the space of all distributions of the form $L+[f]^0$, where $L \in (\mathscr{L})$ and $f() \in \mathscr{F}_-$.

1.18. Lemma. $\mathfrak{B}_- = (\mathscr{L}) + [\mathscr{F}_-]^0$.

PROOF. If A belongs to $(\mathscr{L})+[\mathscr{F}_-]^0$, then the equation

(1) $$A = L+[f]^0$$

holds for some $L\in(\mathscr{L})$ and for some $f(\,)\in\mathscr{F}_-$. Since $f(\,)=0$ on $(0, \infty)$ (by 1.8), it follows from 1.10 and 1.5 that

(2) $$\bigcirc([f]^0)\supset(0, \infty).$$

Since $L\in(\mathscr{L})$ we can apply 1.16 to infer the existence of a number $a<0$ such that

(3) $$\bigcirc(L)\supset(a, \infty).$$

In view of (1), (2) and (3), we may apply 1.6 to obtain

(4) $$\bigcirc(A)\supset(0, \infty):$$

this is obtained by using the fact that $a<0$, which implies the equation $(0, \infty)\cap\cap(a, \infty)=(0, \infty)$.

Thus, A equals $\mathbf{0}$ in $(0, \infty)$: we still have to prove that A equals $[f]^0$ in $(a, -a)$. To that effect, observe that L equals $\mathbf{0}$ in (a, ∞) (by (3) and 1.5); from (1) we therefore infer that A equals $[f]^0$ in (a, ∞). Consequently, A belongs to $\mathfrak{B}_-$ (since A also equals $\mathbf{0}$ in $(0, \infty)$ (see (4) and 1.12)).

To prove the converse, suppose that $A\in\mathfrak{B}_-$. In view of 1.12, this implies (4), the existence of a number $a<0$ and a function $g(\,)$ in $\mathscr{F}$ such that

(5) $$A(\varphi) = [g]^0(\varphi) \quad \big(\text{for } \varphi < (a, -a)\big).$$

Since $\bigcirc(A)\supset(0, -a)$ (by (4) and $a<0$), we have

(6) $$A(\varphi) = 0 \quad \big(\text{for } \varphi < (0, -a)\big).$$

Combining (5) and (6), we see that

$$[g]^0(\varphi) = 0 \quad \big(\text{for } \varphi < (0, -a)\big):$$

that is, $[g]^0$ equals $\mathbf{0}$ in $(0, -a)$; therefore, 1.11 gives

(7) $$g(\,) = 0 \quad \text{on} \quad (0, -a).$$

Let $f(\,)$ be the function defined by

(8) $$f(t) = \begin{cases} g(t) & \text{for } a<t<-a \\ 0 & \text{otherwise.} \end{cases}$$

From (7) and (8) it follows that $f(\,)=0$ on $(0, -a)$; since $f(\,)=0$ on $(-a, \infty)$ (by (8)), we see that $f(\,)=0$ on $(0, \infty)$: therefore,

(9) $$\bigcirc([f]^0)\supset(0, \infty),$$

which (by 1.11) implies that $f(\,)=0$ on $(0, \infty)$; consequently, $f(\,)$ belongs to $\mathscr{F}_-$ and

(10) $$[f]^0\in[\mathscr{F}_-]^0.$$

Next, we set

(11) $$L = A-[f]^0:$$

since $A=L+[f]^0$, our conclusion (namely, that A belongs to the space $(\mathscr{L})+[\mathscr{F}_-]^0$) will be obtained by proving that $L\in(\mathscr{L})$. To that effect, observe that the equations

$$(12) \qquad L(\varphi) = A(\varphi)-[f]^0(\varphi) = [g]^0(\varphi)-[g]^0(\varphi)$$

hold for $\varphi<(a, -a)$ and come from (11) and (5). Since $g(\ )=f(\ )$ on $(a, -a)$ (by (8)), the distribution $[g]^0$ equals $[f]^0$ in $(a, -a)$ (by 1.10), so that $[g]^0(\varphi)=[f]^0(\varphi)$ for $\varphi<(a, -a)$: from (12) it now follows that

$$L(\varphi) = 0 \quad (\text{for } \varphi < (a, -a)),$$

that is,

$$(13) \qquad \bigcirc(L)\supset(a, -a).$$

On the other hand, (11), (4), (9) and 1.6 imply that

$$(14) \qquad \bigcirc(L)\supset\bigcirc(A)\cap\bigcirc([f]^0)\supset(0, \infty).$$

From (13)—(14) and 1.3 it therefore follows that

$$\bigcirc(L)\supset(a, -a)\cup(0, \infty) = (a, \infty),$$

which proves that $\bigcirc(L)\supset(a, \infty)$: therefore, $L\in(\mathscr{L})$ (see 1.15). Since $A=L+[f]^0$ and (10), the distribution A belongs to $(\mathscr{L})+[\mathscr{F}_-]^0$.

1.19. Lemma. *Suppose that $B\in\mathfrak{B}_-$. If $B\in\mathfrak{B}_+$ then $B=\mathbf{0}$.*

Proof. If $B\in\mathfrak{B}_+$ it follows from 1.13 that B equals $\mathbf{0}$ in the interval $(-\infty, 0)$: this means that

$$(1) \qquad B(\varphi) = 0 \quad (\text{for } \varphi < (-\infty, 0)).$$

In view of 1.18, the hypothesis $B\in\mathfrak{B}_-$ implies that the equation

$$(2) \qquad B = L+[f]^0$$

holds for some L in $(\mathscr{L})$ and for some $f(\ )$ in $\mathscr{F}_-$. From (1)—(2) it follows that

$$(3) \qquad [f]^0(\varphi) = -L(\varphi) \quad (\text{for } \varphi < (-\infty, 0)).$$

On the other hand, it follows from $L\in(\mathscr{L})$ and 1.16 the existence of a number $a<0$ such that

$$(4) \qquad L(\varphi) = 0 \quad (\text{for } \varphi < (a, \infty)).$$

Combining (3) and (4):

$$[f]^0(\varphi) = 0 \quad (\text{for } \varphi < (a, 0)),$$

so that $[f]^0$ equals $\mathbf{0}$ in $(a, 0)$, whence $f(\ )=0$ on $(a, 0)$ (by 1.11). But our hypothesis $f(\ )\in\mathscr{F}_-$ implies that $f(\ )=0$ on $(0, \infty)$; therefore, $f(\ )=0$ on (a, ∞), so that

$$(5) \qquad [f]^0(\varphi) = 0 \quad (\text{for } \varphi < a, \infty)).$$

From (5) and (4) we see that $L+[f]^0$ equals $\mathbf{0}$ in (a, ∞); from (2) it therefore follows that

$$\bigcirc(B)\supset(a, \infty);$$

but $\bigcirc(B)\supset(-\infty, 0)$ (from (1)), so that 1.3 now gives

$$\bigcirc(B)\supset(-\infty, 0)\cup(a, \infty) = \mathbf{R}. \tag{6}$$

From (6), 1.5, and 1.1 we see that $B(\varphi)=0$ whenever $\varphi<\mathbf{R}$. If $\varphi(\,)\in D$ then $\varphi<\mathbf{R}$, so that the equation $B(\varphi)=\mathbf{0}(\varphi)$ holds for every $\varphi(\,)$ in D. We have proved that $B=\mathbf{0}$.

1.20. Theorem. *The four spaces $\mathfrak{B}_+$, $(\mathscr{L})$, $\mathfrak{B}_-$, and $\mathfrak{B}$ are linear spaces. To any F in $\mathfrak{B}$ there corresponds a unique pair (F_-, F_+) of distributions such that $F_-\in\mathfrak{B}_-$, $F=F_-+F_+$, and $F_+\in\mathfrak{B}_+$. Moreover,*

$$F\in\mathfrak{B}_+ \Leftrightarrow F_- = \mathbf{0} \Leftrightarrow F = F_+, \tag{1.21}$$

and

$$F\in\mathfrak{B}_- \Leftrightarrow F_+ = \mathbf{0} \Leftrightarrow F = F_-. \tag{1.22}$$

Proof. The linearity of $\mathfrak{B}_+$ and $(\mathscr{L})$ follow directly from 1.13, 1.15, and 1.6; it is now easy to verify that the space $(\mathscr{L})+[\mathscr{F}_-]^0$ is linear; therefore, $\mathfrak{B}_-$ is linear (by 1.18); this in turn implies the linearity of the space $\mathfrak{B}$ (defined by $\mathfrak{B}=\mathfrak{B}_-+\mathfrak{B}_+$). Since $\mathfrak{B}$ consists of distributions of the form $B+R$ with $B\in\mathfrak{B}_-$ and $R\in\mathfrak{B}_+$, the uniqueness of the pair (B, R) will be established by proving that the assumption

$$F_-+F_+ = B+R \tag{1}$$

implies $F_-=B$ and $F_+=R$ (when F_- and B belong to $\mathfrak{B}_-$, and when F_+ and R belong to $\mathfrak{B}_+$). From (1) it follows that F_--B belongs to both $\mathfrak{B}_-$ and $\mathfrak{B}_+$; from 1.19 it therefore follows that $F_--B=\mathbf{0}$, which implies $F_-=B$ and $F_+=R$ (by (1)).

To prove (1.21), note that $F\in\mathfrak{B}_+$ implies $F_-+F_+\in\mathfrak{B}_+$, so that $F_-\in\mathfrak{B}_+$; since $F_-\in\mathfrak{B}_-$ we have $F_-=\mathbf{0}$ (by 1.19): the rest is obvious. The proof of 1.22 is entirely similar.

1.23. Notation. *Let $1_-(\,)$ and $1_+(\,)$ be the functions defined by*

$$1_-(t) = \begin{cases} 1 & \textit{for } t<0 \\ 0 & \textit{for } t\geqq 0 \end{cases} \tag{1.24}$$

and

$$1_+(t) = \begin{cases} 0 & \text{for } t\leqq 0 \\ 1 & \text{for } t>0. \end{cases} \tag{1.25}$$

If $f(\,)\in\mathscr{F}$ we set

$$f_-(\,) = 1_-(\,)f(\,) \quad \text{and} \quad f_+(\,) = 1_+(\,)f(\,). \tag{1.26}$$

1.27. Lemma. *If $f(\,)\in\mathscr{F}$ then $[f]^0\in\mathfrak{B}$,*

$$[f_-]^0 = [f]^0_-, \quad \text{and} \quad [f_+]^0 = [f]^0_+. \tag{1.28}$$

PROOF. From 1.24—1.26 and 1.10 we see that $[f_-]^0 \in \mathfrak{B}$ and $[f_+]^0 \in \mathfrak{B}_+$; from 1.24—1.26 it also follows that $[f]^0 = [f_-]^0 + [f_+]^0$; the conclusions are now immediate from 1.20.

§ 2. The operation $\otimes$

In this section our aim is to define a multiplication $\otimes$ on the linear space $\mathfrak{B}$; its main properties will be established. First, let us set down some notation.

If R is a distribution, its distributional derivative ∂R is defined in the usual way [4, p. 323]. We denote by $1()$ the constant function whose value is the unit 1 (the function $1()$ is defined by $1(t)=1$ for $t \in \mathbf{R}$). Note that $\partial[1]^0 = \mathbf{0}$,

$$\partial[1]^0_- = -\delta, \quad \text{and} \quad \partial[1]^0_+ = \delta, \tag{2.1}$$

where $[1]^0_- = [1_-]^0$ (see 1.28) and δ is the Dirac distribution. As usual, if $f() \in \mathscr{F}$ then $\check{f}()$ is the function defined by $\check{f}(x) = f(-x)$. If T is a distribution, then $\check{T}$ is the distribution (also denoted $T^{\vee}$) defined by

$$\check{T}(\varphi) = T(\check{\varphi}) \quad (\text{for } \varphi() \in D). \tag{2.2}$$

It is easily verified that

$$[\check{f}]^0 = [f]^{0\vee} \quad (\text{for } f() \in \mathscr{F}). \tag{2.3}$$

Clearly,

$$\check{S}^{\vee} = S \quad (\text{for } S \text{ in } D'). \tag{2.4}$$

2.5. Lemma. *Suppose that $a<0$. If $\varphi() \in D$ then $\varphi < (-\infty, -a)$ if (and only if) $\check{\varphi} < (a, \infty)$.*

PROOF. If $\varphi < (-\infty, -a)$ we can use 1.2 to assert that $\varphi()$ vanishes outside of some closed subinterval $[-\lambda, -\mu]$ of $(-\infty, -a)$. Therefore, $-\mu < -a$ and $a < \mu$, so that

$$[\mu, \infty) \subset (a, \infty). \tag{1}$$

If $x > \lambda$ and $t < \mu$ then $-x < -\lambda$ and $-\mu < -t$, which implies that $\varphi(-t) = 0 = \varphi(-x)$ (since $\varphi()$ vanishes outside of the interval $[-\lambda, -\mu]$); consequently, $\check{\varphi}()$ vanishes on the set $(-\infty, \mu) \cup (\lambda, \infty)$: this means that the support of $\check{\varphi}()$ is contained in the interval $[\mu, \lambda]$: the conclusion $\check{\varphi} < (a, \infty)$ now comes from (1) and 1.2.

Conversely, suppose that $\check{\varphi} < (a, \infty)$. From 1.2 it follows that $\check{\varphi}()$ vanishes outside of some interval $[\mu, \lambda] \subset (a, \infty)$. Therefore, $a < \mu$ and $-\mu < -a$, so that

$$(-\infty, -\mu] \subset (-\infty, -a). \tag{2}$$

If $t > -\mu$ and $x < -\lambda$ then $-t < \mu$ and $-x > \lambda$, which implies that $\check{\varphi}(-t) = \check{\varphi}(-x) = 0$ (since $\check{\varphi}()$ vanishes outside of $[\mu, \lambda]$); consequently, $\varphi()$ vanishes on the set $(-\infty, -\lambda) \cup (-\mu, \infty)$; therefore, the support of $\varphi()$ is contained in $[-\lambda, -\mu]$: the conclusion $\varphi < (-\infty, -a)$ is now immediate from (2).

2.6. Theorem. *Suppose that $a<0$. If T is a distribution, then*

$$\bigcirc(T)\supset(a,\infty) \quad \textit{if (and only if)} \quad \bigcirc(\check{T})\supset(-\infty,-a).$$

Proof. If $\bigcirc(T)\supset(a,\infty)$ then

$$T(\varphi_1)=0 \quad (\text{for } \varphi_1<(a,\infty)). \tag{1}$$

If $\varphi<(-\infty,-a)$ then $\check{\varphi}<(a,\infty)$ (by 2.5), so that $T(\check{\varphi})=0$ (by (1)); consequently, $\check{T}(\varphi)=0$ (by 2.2). We have just seen that $\check{T}(\varphi)=0$ whenever $\varphi<(-\infty,-a)$: this means that $\bigcirc(\check{T})\supset(-\infty,-a)$.

Conversely, suppose that $\bigcirc(\check{T})\supset(-\infty,-a)$: if $\varphi<(-\infty,-a)$ then $\check{T}(\varphi)=0$. Thus, by 2.2:

$$T(\check{\varphi})=0 \quad (\text{for } \varphi<(-\infty,-a)). \tag{2}$$

If $\varphi_1<(a,\infty)$ we set $\varphi(\,)=\check{\varphi}_1(\,)$; then

$$\varphi_1(\,)=\check{\varphi}(\,) \tag{3}$$

and $\check{\varphi}<(a,\infty)$, whence $\varphi<(-\infty,-a)$ (by 2.5), and the equations

$$0=T(\check{\varphi})=T(\varphi_1)$$

come directly from (2) and (3). We have just seen that $T(\varphi_1)=0$ whenever $\varphi_1<(a,\infty)$: this means that $\bigcirc(T)\supset(a,\infty)$.

2.7. Lemma. *If S and T are distributions such that*

$$\bigcirc(S)\supset(-\infty,\sigma) \quad \text{and} \quad \bigcirc(T)\supset(-\infty,\tau),$$

then $S*T$ is a distribution such that

$$\bigcirc(S*T)\supset(-\infty,\sigma+\tau); \tag{2.8}$$

moreover,

$$S*T=T*S. \tag{2.9}$$

Proof. Note that both S and T belong to D'_+ (see [8, p. 172]); note also that $\bigcirc(F)$ is the set-theoretic complement of Supp F. Conclusions 2.8 and 2.9 now follows from Théorème XIII in [8, p. 172].

2.10. Definition. *If F_1 and F_2 are distributions such that*

$$\bigcirc(F_k)\supset(a_k,\infty) \quad (\text{for } k=1,2), \tag{2.11}$$

we set

$$F_1 \ast F_2=(\check{F}*\check{F}_2)^{\vee}. \tag{2.12}$$

2.13. Lemma. *If F_1 and F_2 are distributions satisfying* 2.11, *then*

$$\bigcirc(F_1 * F_2) \supset (a_1 + a_2, \infty). \tag{2.14}$$

If $f()$ and $g()$ are in $\mathscr{F}_-$, then the equation

$$f * g(t) = \int_{-\infty}^{\infty} f(t-u) g(u)\, du \quad (\text{for } t \in \mathbf{R})$$

defines a function $f*g()$ in $\mathscr{F}_-$ such that

$$[f]^0 * [g]^0 = [f*g]^0. \tag{2.15}$$

Proof. It is not hard to verify that $*$ is the convolution product as defined in [10, pp. 123—124]; in consequence, 2.14 can be derived from Theorem 5.4—2 in [10, p. 125]; further, 2.15 is proved in [10, pp. 126—127]. The present lemma can be proved directly from 2.12; for example, to establish 2.14, observe that $\bigcirc(\check{F}_k)$ contains the interval $(-\infty, -a_k)$ (by 2.11 and 2.6), so that we may use 2.8 to assert that

$$\bigcirc(\check{F}_1 * \check{F}_2) \supset (-\infty, -a_1 - a_2):$$

conclusion 2.14 now results from one more application of 2.6.

2.16. Lemma. *If $L \in (\mathscr{L})$ and $\bigcirc(R) \supset (0, \infty)$ then $L * R$ and $R * L$ both belong to $(\mathscr{L})$.*

Proof. From 1.16 it follows the existence of a number $a < 0$ such that $\bigcirc(L) \supset$ $\supset (a, \infty)$; we may therefore apply 2.14 to obtain

$$\bigcirc(L * R) \supset (a + 0, \infty),$$

whence the conclusion $L * R \in (\mathscr{L})$ now comes from 1.16; on the other hand, the conclusion $R * L = L * R$ comes from 2.12 and 2.9.

2.17. Definition. *If F and G are distributions, we set*

$$F \otimes G = -F_- \circledast G_- + F_+ * G_+. \tag{2.18}$$

2.19. Theorem. *If F and G belong to $\mathfrak{B}$, then*

$$F \otimes G \text{ belongs to } \mathfrak{B}, \tag{2.20}$$

$$(F \otimes G)_- = -F_- \circledast G_-, \tag{2.21}$$

and

$$(F \otimes G)_+ = F_+ * G_+; \tag{2.22}$$

moreover,

$$F \in \mathfrak{B}_+ \quad \text{implies} \quad F \otimes G = F_+ * G_+ \in \mathfrak{B}_+ \tag{2.23}$$

and

$$F \in (\mathscr{L}) \quad \text{implies} \quad F \otimes G \in (\mathscr{L}). \tag{2.24}$$

PROOF. Clearly,

(1) $$F \otimes G = A + B, \quad \text{where} \quad B = F_+ * G_+$$

and

(2) $$A = -F_- * G_- .$$

A distribution Q belongs to $\mathfrak{B}_+$ if (and only if) $\bigcirc(Q)$ includes the interval $(-\infty, 0)$: see 1.13 and 1.5. Since F_+ and G_+ belong to $\mathfrak{B}_+$, we can use 2.7 to assert that

(3) $$F_+ * G_+ \quad \text{belongs to} \quad \mathfrak{B}_+ .$$

In view of (1)—(3), the conclusion $F \otimes G \in \mathfrak{B}$ can be obtained by proving that

(4) $$F_- * G_- \quad \text{belongs to} \quad \mathfrak{B}_- .$$

Let us prove (4). From 1.18 we see that both F_- and G_+ belong to $(\mathscr{L}) + [\mathscr{F}_-]^0$; therefore, the equations

$$F_- = L^F + [f]^0 \quad \text{and} \quad G_- = L^G + [g]^0$$

hold for $L^F \in (\mathscr{L})$, $L^G \in (\mathscr{L})$, $f(\,) \in \mathscr{F}_-$, and $g(\,) \in \mathscr{F}_-$. Consequently,

(5) $$F_- * G_- = L^F * L^G + L^F * [g]^0 + [f]^0 * L^G + [f]^0 * [g]^0 .$$

The three first terms on the right-hand side of (5) are of the form $R * L$ (or $L * R$), where $\bigcirc(R) \supset (0, \infty)$ and $L \in (\mathscr{L})$; in view of 2.16, their sum is an element L_1 of $(\mathscr{L})$:

(6) $$F_- * G_- = L_1 + [f]^0 * [g]^0 .$$

From (6) and 2.15 it follows that $F_- * G_- = L_1 + [f * g]^0$, where $f * g(\,) \in \mathscr{F}_-$. Therefore, $F_- * G_-$ belongs to the space $(\mathscr{L}) + [F_-]^0$: Conclusion (4) now comes from 1.18.

Having thus proved (4), Conclusions 2.20—2.22 follow directly from (1)—(4) and 1.20. It remains to prove 2.23—2.24. If $F \in \mathfrak{B}_+$ then $F_- = \mathbf{0}$ (by 1.21); consequently, $(F \otimes G)_- = \mathbf{0}$ (by 2.21), which implies $F \otimes G \in \mathfrak{B}_+$ (by 1.21) and $F \otimes G = F_+ * G_+$ (by 2.18). Finally, let us prove 2.24. If $F \in (\mathscr{L})$ then $F \in \mathfrak{B}_-$ (by 1.18), so that $F_+ = \mathbf{0}$ and $F = F_-$ (by 1.22): from 2.18 it therefore follows that

(7) $$F \otimes G = -F * G_- .$$

Our conclusion $F \otimes G \in (\mathscr{L})$ is now obtained by setting $F = L$ and $G_- = R$ in 2.16 (note that $\bigcirc(G_-) \supset (0, \infty)$ since $G_- \in \mathfrak{B}_-$ and 1.12).

2.25. Notations. As indicated at the beginning of this section, the derivative of a distribution F is denoted ∂F (see [4, p. 323]); the Dirac distribution is denoted δ (it is defined by the equation $\delta(\varphi) = \varphi(0)$: see [4, p. 314]).

2.26. Theorem. *If R, S, T belong to $\mathfrak{B}$, then*

$$R \otimes S = S \otimes R, \tag{2.27}$$

$$R \otimes (S \otimes T) = (R \otimes S) \otimes T, \tag{2.28}$$

$$\delta \otimes S = S_+, \tag{2.29}$$

$$\partial(R \otimes S) = -\partial R_- * S_- + \partial R_+ * S_+, \tag{2.30}$$

and

$$\partial([1]^0 \otimes S) = S. \tag{2.31}$$

Proof. Note that $F \in B_+$ if (and only if) F is a distribution whose support is included in the half-open interval $[0, \infty)$ (see 1.13): we may therefore combine Remark 3 in [4, p. 385] with [4, p. 390] and [4, p. 392] to infer that

$$R_+ * S_+ = S_+ * R_+, \tag{1}$$

$$R_+ * (S_+ * T_+) = (R_+ * S_+) * T_+, \tag{2}$$

and

$$\partial(R_+ * S_+) = \partial R_+ * S_+. \tag{3}$$

Since $\bigcirc(F_-) \supset (0, \infty)$ whenever $F \in \mathfrak{B}$, the corresponding equations

$$R_- * S_- = S_- * R_-, \tag{i}$$

$$R_- * (S_- * T_-) = (R_- * S_-) * T_-, \tag{ii}$$

and

$$\partial(R_- * S_-) = \partial R_- * S_- \tag{iii}$$

can be obtained from (1)—(3) by applying 2.12, 2.16, and 2.13 (alternatively, (i)—(iii) can be obtained by verifying that they are consequences of [4, p. 390] and [4, p. 392]). The equation

$$R \otimes S = -S_- * R_- + S_+ * R_+$$

is from 2.18, (1), and (i); another application of 2.18 now gives 2.27. Next, Definition 2.18 gives

$$R \otimes (S \otimes T) = -R_- * (S \otimes T)_- + R_+ * (S \otimes T)_+,$$

so that, by 2.21—2.22:

$$R \otimes (S \otimes T) = R_- * (S_- * T_-) + R_+ * (S_+ * T_+);$$

we may now apply (2) and (ii) to obtain

$$R \otimes (S \otimes T) = (R_- * S_-) * T_- + (R_+ * S_+) * T_+;$$

but 2.21—2.22 then give

$$R \otimes (S \otimes T) = -(R \otimes S)_- * T_- + (R \otimes S)_+ * T_+,$$

and conclusion 2.28 is now immediate from 2.18.

Next, observe that δ belongs to $\mathfrak{B}_+$; therefore, $\delta_- = \mathbf{0}$ and $\delta_+ = \delta$; the equations

$$\delta \otimes S = \delta_+ * S_+ = \delta * S_+ = S_+$$

are from 2.18, from $\delta_+ = \delta$, and from Proposition 9 in [4, p. 391]. We still have to prove 2.30—2.31. From 2.18 it follows immediately that

$$\partial(R \otimes S) = -\partial(R_- * S_-) + \partial(R_+ * S_+):$$

Conclusions 2.30 is now immediate from (3) and (iii); on the other hand, the equations

(4) $$\partial([1]^0 \otimes S) = -\partial[1]^0_- * S + \partial[1]^0_+ * S_+ = \delta * S_- + \delta * S_+$$

are from 2.30 and 2.1; on the other hand, the equations

(5) $$\delta * S_- + \delta * S_+ = S_- + S_+ = S$$

are from Proposition 9 in [4, p. 391] and 1.20. Conclusion 2.31 comes directly from (4)—(5).

2.32. Theorem. *If $f()$ and $g()$ belong to $\mathscr{F}$, then $[f]^0 \otimes [g]^0 = [f \wedge g]^0$, where $f \wedge g()$ is the function in $\mathscr{F}$ defined by*

(2.33) $$f \wedge g(t) = \int_0^t f(t-u) g(u)\, du \quad (\text{for } t \in \mathbf{R}).$$

Proof. From 2.18, 1.28, and 2.15 it follows that $[f]^0 \otimes [g]^0 = [h]^0$, where

(6) $$h() = -f_- * g_-() + f_+ * g_+();$$

the proof will therefore be completed by showing that $h() = f \wedge g()$. To begin with, suppose that $F() \in \mathscr{F}$ and note that the equations

(2.34) $$F_-(t) = \begin{cases} F(t) & \text{for} \quad t < 0 \\ 0 & \text{for} \quad t \geqq 0 \end{cases}$$

and

(2.35) $$F_+(t) = \begin{cases} 0 & \text{for} \quad t \leqq 0 \\ F(t) & \text{for} \quad t > 0 \end{cases}$$

are immediate consequences of 1.24—1.26. Next, it is not hard to verify that $f_+ * g_+() = 0$ on $(-\infty, 0)$; therefore, (6) gives

(7) $$h(t) = -f_- * g_-(t) \quad (\text{for } t < 0).$$

If $t < 0$ then

(8) $$f_- * g_-(t) = \int_{-\infty}^0 f_-(t-u) g(u)\, du = \int_t^0 f(t-u) g(u)\, du:$$

the first equation is from 2.34 (with $F = g$) and the second is obtained by observing that $u > t$ implies $t - u < 0$, whence $f_-(t-u) = f(t-u)$ (by 2.34 with $F = f$), whereas $u < t$ implies $t - u > 0$ and $f_-(t-u) = 0$. From (8) and 2.33 it follows that

$$-f_- * g_-(t) = f \wedge g(t) \quad (\text{for } t < 0);$$

combining with (7): $h(\,)=f\wedge g(\,)$ on $(-\infty, 0)$. Next, to prove the same relation on $(0, \infty)$, take $t>0$: the equations

$$(9) \qquad f_+ * g_+(t) = \int_0^\infty f_+(t-u)g(u)\,du = \int_0^t f(t-u)g(u)\,du$$

come from 2.35 (with $F=g$) and from the fact that $f_+(t-u)=0$ for $u>t$ (see 2.35 with $F=f$). From (9) and 2.33 it follows that

$$(10) \qquad f_+ * g_+(t) = f\wedge g(t) \quad (\text{for } t>0).$$

It is not hard to verify that $f_- * g_-(t)=0$ (for $t>0$); consequently, (6) gives

$$h(t) = f_+ * g_+(t) \quad (\text{for } t>0).$$

Combining with (10), we obtain: $h(\,)=f\wedge g(\,)$ on $(0, \infty)$; since we have already verified that $h(\,)=f\wedge g(\,)$ on $(-\infty, 0)$, we have concluded the proof.

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