On a matrix equation defined over a commutative Euclidean ring

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1. In his paper [1] the author proved the following theorem:

Theorem 1.1. *Let A, B be $m\times m$ and $n\times n$ matrices respectively defined over the complex field. Necessary and sufficient conditions that A, B have* 0 *as an eigenvalue of multiplicity 1 with $Aa_0=Bb_0=0$, are that two nonzero vectors a (of dimension m) and b (of dimension n) exist such that the general solution of the matrix equation $AXB=0$ is given by the expression $X=\gamma a_0 b_0^*+\Lambda b_0^*+a_0\omega^*$, where the parameters are γ, Λ, ω and Λ, ω run through all vectors satisfying $a^*\Lambda=b^*\omega=0$.*

We denoted here the transpose of a matrix, thus the row vectors by $*$.

In this paper partly we generalize this theorem partly we extend the validity of this theorem for matrices defined over a commutative Euclidean ring.

2. Let A, B be $m\times m$ and $n\times n$ matrices respectively defined over the commutative Euclidean ring $\mathscr{R}$. Let $m-r$, $n-r$ be the ranks of the matrices A and B respectively. Then ([2], 401, Satz 233) there exist invertible quadratic matrices P, R, T, S defined over $\mathscr{R}$ such that the equalities

$$PAR=\begin{pmatrix}0\dots 0 & 0\dots 0\\ \dots\dots & \dots\dots\\ 0\dots 0 & 0\dots 0\\ 0\dots 0 & e_1\dots 0\\ \dots\dots & \dots\dots\\ 0\dots 0 & 0\dots e_{m-r}\end{pmatrix}=A_1, \tag{1}$$

$$SBT=\begin{pmatrix}0\dots 0 & 0\dots 0\\ \dots\dots & \dots\dots\\ 0\dots 0 & 0\dots 0\\ 0\dots 0 & e_1'\dots 0\\ \dots\dots & \dots\dots\\ 0\dots 0 & 0\dots e_{n-s}'\end{pmatrix}=B_1 \tag{2}$$

hold. Here $e_1, \dots, e_{m-r}$ and $e_1', \dots, e_{n-s}'$ are the elementary divisors of the matrices A and B respectively. (1) and (2) are the so called normalform of the corresponding matrices.

Let

$$R = (R_{mr} R_{m\,m-r}), \cdot S = \begin{pmatrix} S_{sn} \\ S_{n-sn} \end{pmatrix}$$

be a partition of R and S respectively, where the first and second indices denote the number of the rows and of the columns respectively.

Theorem 2. 1. *Let A, B be $m\times m$ and $n\times n$ matrices with normalforms* (1) *and* (2) *respectively defined over the commutative Euclidean ring $\mathcal{R}$. Necessary and sufficient conditions that A, B have ranks $m-r$ $(r=0, 1, \ldots, m)$ and $n-s$ $(s=0, 1, \ldots, n)$ respectively that the general solution of the matrix equation $AXB=0$ is given by the expression*

$$X = R_{mr}\gamma_{rs}S_{sn} + \Lambda_{ms}S_{sn} + R_{mr}\omega_{rn}, \tag{3}$$

where γ_{rs}, Λ_{ms}, ω_{rn} are parameters defined over $\mathcal{R}$ on condition that if we complete $R_{m\,m-r}$ by an arbitrary column of Λ_{ms} and $S_{n-s\,n}$ by an arbitrary row of ω_{rn} respectively we obtain matrices with ranks $m-r$ and $n-s$ respectively.

If $r=0$ or $s=0$ then the corresponding members of (3) are equal to the zero matrix.

Proof. Obviously that the equation $AXB=0$ is equivalent to

$$(PAR)(R^{-1}XS^{-1})(SBT) = 0$$

and therefore to the equation $A_1YB_1=0$, where $Y=R^{-1}XS^{-1}$ and A_1, B_1 are defined by the formulas (1) and (2) respectively. In consequence of these as the general solution of $A_1YB_1=0$ is given by the expression

$$Y = \begin{pmatrix} \gamma_{rs} & b_{r\,n-s} \\ a_{m-r\,s} & (0) \end{pmatrix}, \tag{4}$$

where γ_{rs}, $a_{m-r\,s}$, $b_{r\,n-s}$ are parameters defined over $\mathcal{R}$, if and only if A_1 and B_1 have ranks $m-r$ and $n-s$ respectively. We obtained consequently:

Necessary and sufficient condition that A, B have rank $m-r$ and $n-s$ respectively that the general solution of the matrix equation $AXB=0$ is given by $X=RYS$ where Y is equal to the matrix (4).

It remains to show that the solution $X=RYS$ is equivalent to the expression (3).

If we introduce the following notations

$$Y_1 = \begin{pmatrix} \gamma_{rs} & (0) \\ (0) & (0) \end{pmatrix}, \quad Y_2 = \begin{pmatrix} (0) & (0) \\ a_{m-r\,s} & (0) \end{pmatrix}, \quad Y_3 = \begin{pmatrix} (0) & b_{r\,n-s} \\ (0) & (0) \end{pmatrix},$$

then $Y=Y_1+Y_2+Y_3$ and therefore we get

$$X = RYS = RY_1S + RY_2S + RY_3S. \tag{5}$$

Considering that

$$RY_1S = R_{m\,r}\gamma_{rs}S_{sn}, \quad RY_2S = P_{m\,m-r}a_{m-r\,s}S_{sn}, \quad RY_3S = P_{mr}b_{r\,n-s}S_{n-s\,n}$$

and that by the notations

$$P_{m\,m-r}a_{m-r\,s} = \Lambda_{ms}, \quad b_{r\,n-s}S_{n-s\,n} = \omega_{rn}, \tag{6}$$

we obtain (3) from the expression (5). Conversely we come to (5) from (3) only if the equations (6) are compatible for given $\Lambda_{ms}, \omega_{rn}$ and for the unknown a_{m-rs}, $b_{r\,n-s}$ respectively. Necessary and sufficient condition of these is that if we complete $R_{m\,m-r}$ by an arbitrary column of Λ_{ms}, and $S_{n-s\,n}$ by an arbitrary row of ω_{rn} respectively then let us have matrices with ranks of $R_{m\,m-r}$ and $S_{n-s\,n}$ respectively. Considering that R and S are invertible, the matrices $R_{m\,m-r}$, $S_{n-s\,n}$ have ranks $m-r$, $n-s$ respectively. This completes the proof of the theorem 2.1.

Theorem 2.2. *Let A, B, C be $m\times m$, $n\times n$, and $m\times n$ matrices respectively defined over the commutative Euclidean ring $\mathcal{R}$. Let $m-r$, $n-s$ and* (1), (2) *be the ranks and the normalform of A, B respectively. The inhomogeneous equation*

$$AXB = C \tag{7}$$

is compatible if and only if the matrix C has the form

$$C = P^{-1}A_1C_1B_1T^{-1}, \tag{8}$$

where C_1 is an arbitrary matrix defined over $\mathcal{R}$. In this case the general solution of the equation (7) *is given by the expression*

$$Z = RC_1S+X, \tag{9}$$

where X is the general solution (3) *of the homogeneous equation $AXB=0$.*

Proof. Considering that (7) is equivalent to the equation $P^{-1}A_1YB_1T^{-1}$, the existence of (8) is necessary for the compatibility of (7). But it is also sufficient. Namely in the case (8) the equation (7) is satisfied by

$$Z_1 = RC_1S. \tag{10}$$

Indeed, taking (7) into account we get

$$AZ_1B = P^{-1}(PAR)R^{-1}RC_1SS^{-1}(SBT)T^{-1} = P^{-1}A_1C_1B_1T^{-1} = C. \tag{11}$$

Denote Z an arbitrary solution of the equation (7). According to (11) the matrix (10) satisfies also the equation (7), therefore $X=Z-Z_1$ is a solution of the homogeneous equation $AXB=0$. Taking into consideration that the general solution of this homogeneous equation is given by the expression (3) we get that the general solution of (7) is the matrix (9) indeed.

References

[1] B. Gyires, A question about the randomized blocks. *Colloquium Mathematicum Societatis János Bolyai. 9. European Meeting of Statisticians.* I. (1974), 277—288.
[2] L. Rédey, Algebra. I. *Leipzig*, 1959.

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