Characterizations of the Baer radical class by almost nilpotent rings

By RICHARD WIEGANDT (Budapest)

To the memory of my beloved friend Prof. A. Kertész

All rings considered, are associative. A radical class will always mean a radical class in the sense of Kurosh and Amitsur. For the terminology and basic results we refer to [1] and [3]. In [2] VAN LEEUWEN and HEYMAN introduced and investigated almost nilpotent rings. A ring A is called an *almost nilpotent* one, if every proper homomorphic image of A is nilpotent.*) It has been proved in [2] that an almost nilpotent ring is either a Baer radical ring or Baer semisimple. In this note we generalize this statement and give two characterizations of the class of Baer (lower) radical rings (Corollaries 1 and 2).**)

In what follows $\mathbf{L}$ and $\mathbf{O}$ will denote the class of all almost nilpotent rings and that of one element rings, respectively. Let us recall that the class $\mathbf{B}$ of all Baer radical rings is just the lower radical class $\mathscr{L}\mathbf{Z}$ of the class $\mathbf{Z}$ of all zero rings, further $\mathbf{B}$ is the upper radical class $\mathscr{U}\mathbf{P}$ of the class $\mathbf{P}$ of all prime rings. For any radical class $\mathbf{R}$ the class of all $\mathbf{R}$-semisimple rings will be denoted by $\mathscr{S}\mathbf{R}$.

Let $\mathbf{R}$ be a radical class and consider the following condition:

(L) *Any almost nilpotent ring A is either $\mathbf{R}$-radical or $\mathbf{R}$-semisimple.*

Theorem 1. *Let $\mathbf{R}$ be a radical class such that $\mathbf{R}\cap\mathbf{L}\neq\mathbf{O}$. $\mathbf{R}$ satisfies condition* (L) *iff $\mathbf{R}$ contains the class $\mathbf{Z}$ of all zero rings.*

PROOF. Assume $\mathbf{Z}\subseteq\mathbf{R}$. In this case the class $\mathbf{R}$ contains all nilpotent rings, too. Let A be an almost nilpotent ring. If I is an ideal of A such that $0\neq I\neq A$, then A/I is nilpotent and so I is not a prime ideal of. A. Hence either A has no prime ideals or only 0 is a prime ideal of A. In the first case we have $A\in\mathbf{B}=\mathscr{L}\mathbf{Z}\subseteq\mathbf{R}$. In the second case $A\in\mathscr{S}\mathbf{B}$ holds, and we have $A/\mathbf{R}(A)\in\mathscr{S}\mathbf{R}\subseteq\mathscr{S}\mathbf{B}$. Hence either $A\in\mathbf{R}$ or $A/\mathbf{R}(A)$ is not nilpotent. In the latter case since $A\in\mathbf{L}$, it follows $\mathbf{R}(A)=0$ i.e. $A\in\mathscr{S}\mathbf{R}$. Thus condition (L) is established.

Conversely, suppose that the radical class $\mathbf{R}$ does not contain all zero rings. Then there exists a zero ring Z such that $Y=Z/\mathbf{R}(Z)\neq 0$, moreover, $Y\in\mathscr{S}\mathbf{R}$. Since every semisimple class is hereditary and Y is a zero ring, so $\mathscr{S}\mathbf{R}$ contains a zero

*) We mean that each simple ring is almost nilpotent.

**) The author is indebted to DR. B. J. GARDNER (Hobart) for his critical remarks.

ring $Z(n)$ over the cyclic group $C(n)$ for some $n=2, 3, \ldots, \infty$. If $\mathbf{R}$ contains a zero ring $X\neq 0$, then the direct sum $A=X\oplus Z(n)$ is almost nilpotent (in fact a zero ring) such that $0\neq X=\mathbf{R}(A)\neq A$ and condition (L) is not satisfied. If $\mathbf{R}\cap\mathbf{Z}=\mathbf{O}$, then clearly $\mathscr{S}\mathbf{R}$ contains all nilpotent rings. Since $\mathbf{R}\cap\mathbf{L}\neq\mathbf{O}$, there exists a ring $A\neq 0$ in $\mathbf{R}\cap\mathbf{L}$. A can be decomposed into a subdirect sum $A\cong \sum_{\alpha}' A_\alpha$ of subdirectly irreducible rings A_α. Taking into account that A is almost nilpotent, either each A_α is nilpotent or $A\cong A_\alpha$ is subdirectly irreducible. The first case is not possible because $\mathbf{R}$ is homomorphically closed and $\mathscr{S}\mathbf{R}$ contains all nilpotent rings. Hence A is subdirectly irreducible. Since $\mathscr{S}\mathbf{R}$ contains all nilpotent rings, by $0\neq A\in\mathbf{R}$ it follows that A is not nilpotent. Denoting the heart of A by H and taking into account $A\in\mathbf{L}$, we have $0\neq(A^n)^2\subseteq H^2$. Consequently H is an idempotent simple ring. We exhibit $H\in\mathbf{R}$. In the case $H\in\mathscr{S}\mathbf{R}$ being A/H nilpotent, we get $A/H\in\mathscr{S}\mathbf{R}$. Taking into account that every semisimple class is closed under forming extensions, it follows $A\in\mathscr{S}\mathbf{R}$ contradicting $0\neq A\in\mathbf{R}$. Hence $H\in\mathbf{R}$ holds. Consider the ring B defined as follows:

$$B^+ = H^+\oplus C^+$$

where C^+ is the cyclic additive group $C(p)$ if H has characteristic p and C^+ denotes the additive group Q^+ of rationals if H has characteristic 0. Further, the product of any two elements (k_1, c_1) and (k_2, c_2) of B should be defined by

$$(k_1, c_1)(k_2, c_2) = (k_1k_2+c_2k_1+c_1k_2, 0).$$

Then B is a ring, moreover, B is obviously almost nilpotent and $0\neq H=\mathbf{R}(B)\neq B$. Thus the almost nilpotent ring B does not satisfy condition (L).

Corollary 1. *A radical class* $\mathbf{R}$ *coincides with the class* $\mathbf{B}$ *of Baer radical rings iff*

(i) $\mathbf{R}$ *satisfies condition* (L);

(ii) $\mathbf{R}\cap\mathbf{L}\neq\mathbf{O}$;

(iii) *for any radical class* $\mathbf{R}'$ *with properties* (i) *and* (ii) *the inclusion* $\mathbf{R}\subseteq\mathbf{R}'$ *holds.*

The assertion is obvious in view of Theorem 1 and $\mathbf{B}=\mathscr{L}\mathbf{Z}$.

The next nexample shows that condition (ii) cannot be omitted from Corollary 1. Let V be a vector space of dimension $\aleph_{\omega_0}$ over a division ring D and let W denote the ring of all linear transformations of V. As it is known (cf. Example 11 of [1] on pp. 109—110) the proper ideals of W are the following ones:

$$I_n = \{A\in W \mid \dim \operatorname{Im} A < \aleph_n\}$$

for $n=0, 1, 2, \ldots, \omega_0$. Consider the class $\mathbf{W}$ of all homomorphic images of the ring W. Obviously $\mathbf{W}$ consists of rings with unity and none of them is simple. Take the lower radical class $\mathscr{L}\mathbf{W}$ determined by the class $\mathbf{W}$. Taking into consideration that the class $\mathbf{L}$ of almost nilpotent rings is hereditary (cf. [2] Lemma) and that the semisimple class $\mathscr{S}\mathscr{L}\mathbf{W}$ contains all nilpotent rings as well as simple rings, it follows $\mathbf{L}\subseteq\mathscr{S}\mathscr{L}\mathbf{W}$. Thus $\mathscr{L}\mathbf{W}\cap\mathbf{L}=\mathbf{O}$ though $\mathscr{L}\mathbf{W}\neq\mathbf{O}$ and $\mathscr{L}\mathbf{W}$ satisfies condition (L).

Confining ourselves to hereditary radicals, condition (ii) can be eliminated from Corollary 1.

Theorem 2. *A hereditary radical class* $\mathbf{R}\neq\mathbf{O}$ *satisfies condition* (L) *iff* $\mathbf{Z}\subseteq\mathbf{R}$.

PROOF. By Theorem 1 $\mathbf{Z} \subseteqq \mathbf{R}$ implies that $\mathbf{R}$ satisfies condition (L). Suppose $\mathbf{Z} \not\subseteqq \mathbf{R}$. If $\mathbf{R}$ contains a zero ring $X \neq 0$ then, as we have already seen in the proof of Theorem 1, condition (L) is not fulfilled. If all zero rings and so all nilpotent rings are contained in $\mathscr{S}\mathbf{R}$ then, taking into consideration $\mathbf{R} \neq \mathbf{O}$ and that $\mathscr{S}\mathbf{R}$ is closed under forming subdirect sums, $\mathscr{S}\mathbf{R}$ does not contain all subdirectly irreducible rings. Hence there exists a subdirectly irreducible ring A such that for the heart H of A the relation $0 \neq H \leqq \mathbf{R}(A)$ holds. Since $\mathbf{R}$ is hereditary, it follows $H \in \mathbf{R}$. Take the direct sum $B = H \oplus Z(p)$ where $Z(p)$ is the cyclic zero ring of prime order p. B is neither in $\mathbf{R}$ nor in $\mathscr{S}\mathbf{R}$, though B is almost nilpotent. Thus condition (L) is not satisfied.

Corollary 2. *A hereditary radical class* $\mathbf{R}$ *coincides with the class* $\mathbf{B}$ *of Baer radical rings iff*

(i) $\mathbf{R}$ *satisfies condition* (L)

(ii) *for any hereditary radical class* $\mathbf{R}'$ *satisfying condition* (L) *the relation* $\mathbf{R} \subseteqq \mathbf{R}'$ *holds.*

Corollary 3. *A radical class* $\mathbf{R} \neq \mathbf{O}$ *satisfying condition* (L) *but not containing the class* $\mathbf{Z}$, *is never hereditary.*

References

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METHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15.
H1053 BUDAPEST, HUNGARY

(Received June 5, 1974.)