Contributions to the ideal theory of semigroups

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To the memory of Professor Andor Kertész

1. Introduction

In Sections 2—4 of this note we discuss some connections between various classes of ideals in semigroups. These investigations will be completed in Section 5 by examples showing that several implications concerning the conditions discussed in the preceding sections are not true in general.

We begin with recalling that an ideal I of a semigroup S is said to be
prime if $ab \in I$ implies $a \in I$ or $b \in I$,
semiprime if $a^2 \in I$ implies $a \in I$,
categorical if $abc \in I$ implies $ab \in I$ or $bc \in I$,
inclusive if $SaS \subseteqq I$ implies $a \in I$
for any elements a, b, c of S. The ideal I of S is called
maximal if $I \neq S$ but $I \subset A$ implies $A = S$,
weakly prime if $AB \subseteqq I$ implies $A \subseteqq I$ or $B \subseteqq I$
for any ideals A, B of S. Finally, an ideal I is said to be
idempotent if $I^2 = I$ and
reproduced (by S) if $SI = IS = I$.
For other concepts and symbols we refer to [1].

2. Maximal ideals

It is known that a semigroup S with the identity 1 contains a unique maximal ideal, which is the union of all the ideals I of S such that $I \not\ni 1$ (see [6]).

Theorem 1. *Let S be a semigroup with identity and let M denote the maximal ideal of S. Then M is weakly prime. If, moreover,*
(A) *the mapping $a \to J(a)$ $(a \in S)$ is a homomorphism of S, or*
(B) *S has no subsemigroup isomorphic to the bicyclic semigroup,*
then M is prime, too.

COROLLARY. *For any commutative or periodic semigroup S with identity, the unique maximal ideal of S is prime.* (For commutative semigroups, see [6].)

Proof. By a theorem of Grillet ([2], Proposition 1) any maximal ideal M of an arbitrary semigroup S is either weakly prime or of the from $M=S\setminus\{a\}$ with an element a that cannot be decomposed into the product of two factors in S. Clearly, the second case cannot occur when S has an identity. Thus the first assertion of the theorem is proved.

Assuming (A), consider any elements u, $v\in S\setminus M$. Then, by the maximality of M,

$$M\cup J(u) = M\cup J(v) = S\ni 1$$

whence $J(u)\ni 1$, $J(v)\ni 1$ and

$$J(uv) = J(u)\cdot J(v) = SS = S.$$

This implies $uv\notin M$, completing the proof that M is prime in this case.

Assume (B). Since $J(a)=S$ for each $a\in S\setminus M$, the Rees factor semigroup S/M is 0-simple. Then, by a theorem due to O. Andersen (Theorem 2.54 in [1]), S is completely 0-simple, too. Hence, by a theorem of D. Rees ([5]; see also Exercise 1 for § 2.6 in [1]), S/M is a group with zero. Consequently, M is a prime ideal of S.

Corollary follows at once from the facts that every commutative semigroup satisfies (A) (see, e.g., [8], Lemma 7) and every periodic semigroup satisfies (B).

3. Categorical ideals

It is remarked in [3] without proof that *any prime ideal is categorical.* For sake of completeness we give a short proof. Let P be a prime ideal and a, b, c any elements of the semigroup S such that $abc\in P$. Then $ab\in P$ or $c\in P$. But $c\in P$ implies $bc\in P$. Hence, P is categorical, indeed.

Concerning semigroups with identity the converse statement is likewise true:

Theorem 2. *Any categorical ideal of a semigroup with identity is prime.*

Proof. Let C be a categorical ideal, e the identity and a, b any elements of the semigroup S. If $ab\in C$, then $aeb\in C$ and therefore $a=ae\in C$ or $b=eb\in C$.

4. Inclusive ideals

It is remarked in [4] that *any prime ideal is inclusive.* This assertion can be improved as follows:

Theorem 3. *Any semiprime or weakly prime ideal of a semigroup is inclusive.*

Corollary. *There exist inclusive ideals that are not prime.*

Proof. Let A be a semiprime ideal of the semigroup S and $SaS\subseteq A$ for some $a\in S$. Then $a^4=a^2aa\in A$ whence $a^2\in A$ and, further, $a\in A$. This means that A is inclusive.

Let B be a weakly prime ideal of S. Since $SaS=SS^1aS^1S=S\cdot J(a)\cdot S$ for each $a\in S$, any inclusion $SaS\subseteq B$ implies $S\subseteq B$ or $J(a)\subseteq B$, and $a\in B$ in both cases. Thus B is inclusive.

Corollary is trivial.

Theorem 4. *The following conditions concerning a semigroup S are equivalent:*
(A) Every ideal of S is inclusive.
(B) Every ideal of S is reproduced by S.

COROLLARY. *Every ideal of a regular (left regular, right regular or intraregular) semigroup is inclusive.*

PROOF. Assume (B). Then $a \in SaS$ for each $a \in S$ ([7], Corollary 1 to Theorem 3), implying (A) trivially.

In order to show that (A) implies (B), suppose that (B) does not hold. Then there exists a principal ideal $J(a)$ $(a \in S)$, too, which is not reproduced by S and therefore $a \notin SaS$. Consequently, the ideal SaS is not inclusive and (A) does not hold. By this indirect argumentation the proof of the theorem is completed.

Corollary follows immediately by [7], Theorem 4.

5. Additional remarks

In [9] we introduced the following definition:

DEFINITION. Let T be a property concerning the ideals of semigroups. Then the semigroup S is called a *semigroup with T-ideals* if every ideal of S has the property T.

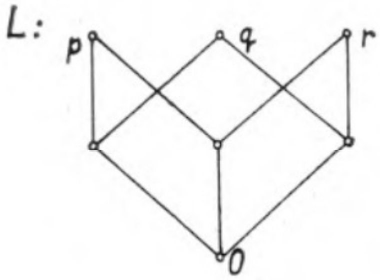
For sake of brevity we shall use also the following notations (where "c.s.w." means: "the class of semigroups with"):

$\mathscr{P}$: c. s. w. prime ideals
$\mathscr{S}$: c. s. w. semiprime ideals
$\mathscr{W}$: c. s. w. weakly prime ideals
$\mathscr{I}$: c. s. w. idempotent ideals
$\mathscr{R}$: c. s. w. reproduced ideals
$\mathscr{C}$: c. s. w. categorical ideals

It is known (see [9]) that

$$(*) \qquad \mathscr{P} = \mathscr{S} \cap \mathscr{W} \subset \mathscr{S} \subset \mathscr{I} \subset \mathscr{R} \quad \text{and} \quad \mathscr{W} \subset \mathscr{I},$$

and $\mathscr{P} \subseteqq \mathscr{C}$ by the first paragraph of Section 3. In order to get more information about the class $\mathscr{C}$, consider the meet-semilattice L given by its diagram and the semigroup T given by its Cayley table as follow:



T	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	0	a	b	0	e	b	a
b	0	a	b	0	0	e	0	a	b
c	0	0	0	c	d	0	f	d	c
d	0	c	d	0	0	f	0	c	d
e	0	a	b	a	b	e	e	e	e
f	0	c	d	c	d	f	f	f	f
g	0	c	d	a	b	f	e	1	g
1	0	a	b	c	d	e	f	g	1

(T is isomorphic to the semigroup of all transformations φ of the set $X=\{0, 1, 2\}$ such that $0\varphi=0$; the ideals of T are: $\{0\}$, T and $T\setminus\{g, 1\}$). Consider, further, the subsemigroup $U=\{0, a, b\}$ of T. It is easy to see that $L\in\mathcal{S}\setminus\mathcal{C}$ and $T\in\mathcal{W}\setminus\mathcal{C}$ (in both cases the zero ideal is not categorical, see the products pqr and agc, respectively). Moreover, $U\in\mathcal{C}\setminus\mathcal{R}$. This means that *for the class $\mathcal{C}$ of semigroups none of the inclusions $\mathcal{S}\subseteq\mathcal{C}$, $\mathcal{W}\subseteq\mathcal{C}$ and $\mathcal{C}\subseteq\mathcal{R}$ holds*. It follows, by $(*)$, that $\mathcal{I}, \mathcal{R}\not\subseteq\mathcal{S}$ and $\mathcal{C}\not\subseteq\mathcal{I}, \mathcal{S}, \mathcal{W}, \mathcal{P}$. Especially, *the term "with identity" cannot be deleted from the text of Theorem* 2.

Consider now the subsemigroup $Z=\{0, a\}$ of T. Clearly, Z is an inclusive ideal of itself, but it is not reproduced by Z. On the other hand, the zero ideal of Z is reproduced by Z (what is more, it is idempotent), but it is not inclusive. Hence, *inclusive ideals are not necessarily reproduced, and conversely.*

Finally we remark that, without assuming the existence of an identity, Theorem 1 does not remain valid in general. In fact, by considering the zero ideal of Z one can see that *the condition that a semigroup has a unique maximal ideal does not imply that this maximal ideal is weakly prime.*

References

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