A note on primitive classes of arithmetic rings

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Dedicated to the memory of my beloved friend Prof. Andor Kertész

1. In a recent paper [4] MICHLER and WILLE have shown: a primitive class **R** of (associative) rings consists of arithmetic rings if and only if **R** is generated by a finite set S of finite fields.

A class **R** of (associative) rings is primitive if **R** is closed with respect to subrings, epimorphic images and direct products, and a ring R is arithmetic if the lattice of its two-sided ideals is distributive. **R** is said to be the primitive class generated by S if **R** is the smallest primitive class containing S.

It is the purpose of this note to investigate the character of the rings which belong to primitive classes of arithmetic rings.

A class of rings is called a *radical-semisimple class* if it is a radical class for some radical as well as a semisimple class for another radical. Radical-semisimple classes are completely described by STEWART [6]. WIEGANDT showed that a semisimple class is homomorphically closed if and only if it is a radical class. Let $\mathbf{K}_n$ be the class of all associative rings A such that $x^n=x$ for each $x\in A$ $(n=2, 3, \dots)$. Then examples of radical semisimple classes are the classes $\mathbf{K}_n$ for $n=2, 3, \dots$, [7], [8]. In any class $\mathbf{K}_n$ there is only a finite set F_n of finite fields and $A\in\mathbf{K}_n$ if and only if A is subdirect sum of fields $F\in F_n$, [3].

We show that a primitive class **R** of rings consists of arithmetic rings if and only if **R** is a radical semisimple class. So examples of primitive classes of arithmetic rings are the classes $\mathbf{K}_n$ for $n=2, 3, \dots$.

2. A class **R** of rings is closed under extensions if $K/A\in\mathbf{R}$, $A\in\mathbf{R}\Rightarrow K\in\mathbf{R}$ whenever A is an ideal in K.

Now we have.

Theorem 1. *Let* **R** *be a primitive class of rings, where* **R** *is not the class of all rings. Then the following are equivalent:*

(i) **R** *is closed under extensions*
(ii) **R** *is a radical class*
(iii) **R** *is a semisimple class*
(iv) **R** *is a radical-semisimple class*
(v) **R** *is a class of arithmetic rings.*

PROOF. (i)⇒(ii). (Theorem 34.1, [9]).

(ii)⇒(iii). Suppose **R** is a radical class. Then **R** is closed under complete direct sums and **R** is strongly hereditary imply that **R** is closed under subdirect sums. Hence **R** is a semisimple class ([1]).

(iii)⇒(iv). A homomorphically closed semisimple class is a radical-semisimple class ([9]).

(iv)⇒(v). Let $\mathbf{D}=\{A|[a]=[a]^2$ for each $a\in A$, A a ring$\}$, where $[a]$ is the subring of A, generated by a. Then if $\mathbf{R}\subsetneqq\mathbf{D}$, then **R** is the class of all rings. ([6]). Hence $\mathbf{R}\subseteq\mathbf{D}$. Now we show that any ring R in **R** is an arithmetic ring. Let A, B, C be ideals in R and let $a\in A\cap(B+C)$, $a=b+c$ say, $b\in B$, $c\in C$. Then $a^2=ab+ac\in(A\cap B)+(A\cap C)$. Cleary $a^n\in(A\cap B)+(A\cap C)$ for any $n=2, 3, \ldots$. Now $[a]=[a]^2$ implies $a\in[a]^2$ i.e. a can be expressed as a finite sum $a=\sum_{i=2}^{r} k_i a^i$, $k_i\in\mathbf{Z}\Rightarrow a\in(A\cap B)+(A\cap C)$.

This implies that $A\cap(B+C)=(A\cap B)+(A\cap C)$.

Hence R is an arithmetic ring.

(v)⇒(i). Suppose $S=K/A\in\mathbf{R}$, $A\in\mathbf{R}$, where A is an ideal in K. Since **R** is a primitive class of arithmetic rings, it contains only a finite number of fields ([4], Hilfssatz 3), $\{\mathbf{Z}_{p_1}^{\alpha_1}, \ldots, \mathbf{Z}_{p_n}^{\alpha_n}\}=F$ say.

Any ring in **R** is a subdirect sum of finite fields from F([4], Hilfssatz 4). Conversely, let R be a subdirect sum of finite fields from F. Since **R** is closed under subdirect sums (as a primitive class), $R\in\mathbf{R}$.

Hence $\mathbf{R}=\{A|A$ is a subdirect sum of fields from $F\}$. Then S is a subdirect sum of fields $\{\mathbf{Z}_{p_i}^{\alpha_i}\}$, A is a subdirect sum of rings $\{\mathbf{Z}_{p_j}^{\alpha_j}\}$, where each $\mathbf{Z}_{p_j}^{\alpha_j}$ has an identity.

Then K is a subdirect sum of fields $\{\mathbf{Z}_{p_i}^{\alpha_i}\}\cup\{Z_{p_j}^{\alpha_j}\}\subseteq F$ ([5]), hence $K\in\mathbf{R}$.

Thus **R** has the extension property.

Note that any class $\mathbf{K}_n$ is a primitive class.

The fields of a given $\mathbf{K}_n$ are exactly those in the corresponding F_n, where $R\in\mathbf{K}_n$ if and only if R is a subdirect sum of fields from F_n. It may be pointed out that for $n\neq m$, $F_n=F_m$ is possible and hence $\mathbf{K}_n=\mathbf{K}_m$, not as asserted in [7] Satz 1, 3. For example,

$$F_4 = \{\mathbf{Z}_2, \mathbf{Z}_{2^2}\} = F_{10} \quad \text{(cf. [3])}.$$

Since the $\mathbf{K}_n$ are strongly hereditary classes, every F_n is necessarily a strongly hereditary finite set of finite fields. However, not every strongly hereditary finite set of finite fields is some F_n, for $\{\mathbf{Z}_2, \mathbf{Z}_{2^2}, \mathbf{Z}_3\}$ is such a set and is not equal to any F_n and is a proper subset of F_{13}. The precise result is:

a finite field $\mathbf{Z}_p^k\in F_n$ if and only if p^k-1 is a divisor of $n-1$ ([2], Lemma 2).

Remark. JIANG LUH has shown that each ring in $\mathbf{K}_n$ is a direct sum of finitely many p^k-rings (p a prime, k a positive integer). A ring R is called a p^k-ring if there exist a prime p and a positive integer k such that $x^{p^k}=x$ and $px=0$ for every $x\in R$, [2].

Literature

[1] E. P. ARMENDARIZ, Closure properties in radical theory, *Pac. J. Math.* **26** (1968), 1—7.
[2] LUH, JIANG, On the structure of J-rings, *Amer, Math. Monthly* **74** (1967), 164—166.
[3] L. C. A. VAN LEEUWEN, and T. L. JENKINS, A note on radical semisimple classes, *Publ. Math.* (Debrecen) **21** (1974), 179—184.
[4] G. MICHLER and R. WILLE, Die primitiven Klassen arithmetischer Ringe, *Math. Z.* **113** (1970), 369—372.
[5] R. L. SNIDER, Subdirect decompositions of extension rings, *Mich. Math. Journal,* **16** (1969), 225—226.
[6] P. N. STEWART, Semisimple radical classes, *Pac. J. Math.* **32** (1970), 249—254.
[7] F. A. SZÁSZ, Beiträge zur Radikaltheorie der Ringe, *Publ. Math. (Debrecen)* **17** (1970), 267—272.
[8] R. WIEGANDT, Homomorphically closed semisimple classes, *Studia Univ. Babes—Bolyai, Cluj, Ser. Math.-Mech.,* **17**, (2) (1972), 17—20.
[9] R. WIEGANDT, Radical and semisimple classes of rings, *Queeen's papers in pure and applied mathematics,* no. **37**, *Queen's University Kingston, Ontario, Canada* (1974).

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