Prime ideals and zero-divisors in Noetherian-like rings

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Dedicated to the memory of Professor Andor Kertész

1. Introduction. *Let R be a commutative ring with unit. The zero-divisors of a* non-zero R-module M will be denoted by $Z(M)$, i.e. $Z(M)=\{r\in R/\text{there exists } 0\neq m\in M \text{ with } rm=0\}$. In [4] E. G. EVANS calls R a zero-divisor ring (Z.D.ring) if $Z(R/I)$ is a finite union of prime ideals for each proper ideal I of R. He demonstrates that every non-zero finitely generated module M over a Z.D.ring has the following property:

If I is a finitely generated ideal of R contained in Z(M), then I is the annihilator of some non-zero element of M.

In general, any R-module with this property is called a *pseudo-Noetherian* module. The class of commutative rings determined by the following definition is examined in [7], [8] and [9].

Definition. A *pseudo-Noetherian ring* is a coherent ring which has the property that all of its non-zero finitely presented modules are pseudo-Noetherian.

These rings are interesting primarily because much of the theory of depth and R-sequences developed for local Noetherian rings in [1] and [2] may be extended to local *) pseudo-Noetherian rings.

It is evident from the above remarks that a coherent Z.D.ring is pseudo-Noetherian. The converse is not necessarily true. For example, a Von Neumann regular ring with infinitely many prime ideals is pseudo-Noetherian but not Z.D. The purpose of this paper is to exhibit a *local* pseudo-Noetherian ring which is not a Z.D.ring.

2. The Example. Let $\mathbf{N}$ represent the set of positive integers. Suppose $\{x_n|n\in\mathbf{N}\}$ is an infinite set of indeterminates and K is a field. Denote by R the subring of $K[[x_n|\,n\in\mathbf{N}]]$ consisting of all those power series whose expansions contain only finitely many indeterminates.

CLAIM. R is a local pseudo-Noetherian ring which is not a Z.D. ring.

(i) R is a pseudo-Noetherian ring.

*) By "local" ring we mean a possibly non-Noetherian ring with a unique maximal ideal.

For each $n\in\mathbf{N}$ let $R_n=K[[x_1, x_2, \ldots, x_n]]$ and notice that R is the union of the chain of rings $(R_n)_{n\in\mathbf{N}}$. If $n'\geqq n$, $R_{n'}$ is isomorphic to a direct product of copies of R_n and hence, since R_n is Noetherian, $R_{n'}$ is a flat R_n-module [3, Theorem 2.1]. Furthermore, since the inclusion $R_n\subseteqq R_{n'}$ is local, $R_{n'}$ is a faithfully flat R_n-module [6, Section 4. A]. In [8] it is shown that a directed union of pseudo-Noetherian domains, in which all inclusions are faithfully flat, is a pseudo-Noetherian ring.

For every $n\in\mathbf{N}$ each element $r\in R$ has a unique decomposition $r=\varphi_n(r)+$ $+x_n\psi_n(r)$ where $\varphi_n(r)$ is that portion of r which is not divisible by x_n and $x_n\psi_n(r)$ is the remainder. The map $\varphi_n: r\to\varphi_n(r)$ is an idempotent ring endomorphism of R for each $n\in\mathbf{N}$. These decompositions and endomorphisms are useful tools in the following considerations.

To prove that R is not a Z.D.ring, it is necessary to show that there exists an ideal I of R with the property that $Z(R/I)$ is not a finite union of prime ideals. The idea for the proof is derived from a paper of W. HEINZER and J. OHM in which it is demonstrated that a ring R is Noetherian if (and only if) $R[x]$ is a Z.D. ring [5]. Consider the following polynomials defined by iteration.

$$f_0 = x_2$$
$$f_n = x_1+f_0 f_1\ldots f_{n-1} \quad (n\in\mathbf{N}).$$

Let I represent the ideal of R generated by $\{x_{n+2}f_1 f_2\ldots f_n | n\in\mathbf{N}\}$. It will be shown that $Z(R/I)$ has the desired property.

(ii) $\{f_n|n\in\mathbf{N}\}\subseteqq Z(R/I)$.

Since $I\subseteqq f_1R$, $x_3\notin I$ and hence $f_1\in Z(R/I)$. Now assume $n>1$ and $x_{n+2}f_1 f_2\ldots$ $\ldots f_{n-1}$ is a member of I. Then there exist h_j $(j=1, 2, \ldots, N)$ in R such that $x_{n+2}f_1 f_2\ldots$ $\ldots f_{n-1}=\sum_{j=1}^{N} x_{j+2}f_1 f_2\ldots f_j h_j$. Now apply $\varphi_3\varphi_4\ldots\varphi_{n+1}$ to both sides of this equation and then cancel $f_1 f_2\ldots f_{n-1}$ to obtain $x_{n+2}\in f_nR$. Since this is impossible, $x_{n+2}f_1 f_2\ldots$ $\ldots f_{n-1}\notin I$ and therefore $f_n\in Z(R/I)$.

(iii) $x_1\notin Z(R/I)$

Suppose to the contrary that $x_1\in Z(R/I)$. Among all $h\in R$ with $h\notin I$ but $x_1h\in I$, pick one with a representation $x_1h=\sum_{n=M}^{N} x_{n+2}f_1 f_2\ldots f_n h_n$ of minimum length $N-M$. Since

$$h-\sum_{n=M}^{N} x_{n+2}f_1 f_2 \ldots f_n\psi_1(h_n)\notin I$$

and

$$x_1\left(h-\sum_{n=M}^{N} x_{n+2}f_1 f_2 \ldots f_n\psi_1(h_n)\right) = \sum_{n=M}^{N} x_{n+2}f_1 f_2 \ldots f_n\varphi_1(h_n)\in I$$

we may assume in addition that $\varphi_1(h_n)=h_n$ $(M\leqq n\leqq N)$.

Now

$$x_1 h = x_{M+2}f_1 f_2 \ldots f_M\left(h_M+\sum_{n=M+1}^{N} x_{n+2}f_{M+1}f_{M+2}\ldots f_n\psi_{M+2}(h_n)\right)+$$
$$+\sum_{n=M+1}^{N} x_{n+2}f_1 f_2 \ldots f_n\varphi_{M+2}(h_n).$$

For each n, $M+1\leqq n\leqq N$,

$$f_{M+1}f_{M+2}\cdots f_n=\varphi_1(f_{M+1}f_{M+2}\cdots f_n)+x_1\psi_1(f_{M+1}f_{M+2}\cdots f_n).$$

Therefore

$$x_1\left(h-x_{M+2}f_1f_2\cdots f_M\sum_{n=M+1}^{N}x_{n+2}\psi_1(f_{M+1}f_{M+2}\cdots f_n)\psi_{M+2}(h_n)\right)=$$

$$=x_{M+2}f_1f_2\cdots f_M\left[h_M+\sum_{n=M+1}^{N}x_{n+2}\varphi_1(f_{M+1}f_{M+2}\cdots f_n)\psi_{M+2}(h_n)\right]+$$

$$+\sum_{n=M+1}^{N}x_{n+2}f_1f_2\cdots f_n\varphi_{M+2}(h_n).$$

From above

$$x_1h=\sum_{n=M}^{N}x_{n+2}f_1f_2\cdots f_nh_n\quad\text{with}\quad\varphi_1(h_n)=h_n(M\leqq n\leqq N).$$

Applying φ_1 to both sides of this equation we obtain $0=\sum_{n=M}^{N}x_{n+2}\varphi_1(f_1f_2\ldots f_n)h_n$ and applying φ_{M+2} on this yields

$$0=\sum_{n=M+1}^{N}x_{n+2}\varphi_1(f_1f_2\cdots f_n)\varphi_{M+2}(h_n).$$

Hence,

$$0=x_{M+2}\varphi_1(f_1f_2\cdots f_M)h_M+\sum_{n=M+1}^{N}x_{n+2}\varphi_1(f_1f_2\cdots f_n)x_{M+2}\psi_{M+2}(h_n)$$

i.e.

$$0=x_{M+2}\varphi_1(f_1f_2\cdots f_M)\left[h_M+\sum_{n=M+1}^{N}x_{n+2}\varphi_1(f_{M+1}f_{M+2}\cdots f_n)\psi_{M+2}(h_n)\right].$$

Since $x_{M+2}\varphi_1(f_1f_2\ldots f_M)\neq 0$, the expression in square brackets is zero. Consequently, for

$$h'=h-x_{M+2}f_1f_2\cdots f_M\sum_{n=M+1}^{N}x_{n+2}\psi_1(f_{M+1}f_{M+2}\cdots f_n)\psi_{M+2}(h_n)\quad(h'\notin I)$$

there is a shorter representation

$$x_1h'=\sum_{n=M+1}^{N}x_{n+2}f_1f_2\cdots f_n\varphi_{M+2}(h_n);\ \text{a contradiction.}$$

(iv) $Z(R/I)$ is not a finite union of prime ideals.

If $Z(R/I)$ were a finite union of prime ideals then two different f_m and f_n $(m>n)$ would both lie in one prime ideal. But then $x_1=f_m-f_0f_1\ldots f_n\ldots f_{m-1}$ would also lie in that ideal which contradicts (iii).

Hence, R is not a Z.D. ring.

3. Remark. Let $R=K[x_n|n\in N]_{(x_n\in n|N)}$ be the localization of the polynomial ring in infinitely many indeterminates at the maximal ideal generated by these indeterminates. It is possible to show by a similar argument that R is also a local pseudo-Noetherian ring which is not a Z.D.ring.

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