The groups of homothetic transformations in areal spaces of the submetric class

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Dedicated to the memory of late Prof. Dr. E. T. Davies

§ 1. Introduction

In our previous papers [1, 2] [1]), we have discussed the transformation of curvature tensors in an areal space of the submetric class by means of the conformal transformation, and have investigated several new tensors and their properties. In the mean time, while examining the change of connection parameters under the conformal change, we noticed a particular case in the transformation and subsequently the theory of homothetic transformations was studied and discussed sufficiently in [3] by making the use of the theory of Lie-derivatives. In the present paper, our aim is to discuss the groups of homothetic transformations in areal spaces of the submetric class.

In this paper, we employ the same notations and terminologies as those used in our previous paper [3] without explanations. Moreover, in what follows, the Latin indices $h, i, j, \ldots$ run from 1 to n and Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to m $(1 < m \leqq n)$ throughout this paper.

§ 2. Some preliminary results

In an n-dimensional areal space $A_n^{(m)}$ of submetric class with fundamental function [2]) $F(x, p)$, with the normalized metric tensor $g_{ij}(x, p)$, and equipped with the metric connection $\Gamma^{*}{}_{ij}^{h}(x, p)$ of KAWAGUCHI and TANDAI [7], two kinds of covariant derivations of an absolute contravariant vector $X^i(x, p)$ due to M. GAMA [5] may be defined as follows:

$$X^i_{|k} = X^i_{;k} - X^i;^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{\lambda k} + X^j \Gamma^{*}{}^{i}_{jk}, \tag{2.1}$$

$$X^i|^{\lambda}_{\gamma} = X^i;^{\lambda}_{\gamma} + X^j C^i_{j},^{\lambda}_{\gamma}, \tag{2.2}$$

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1) Numbers in brackets refer to the references at the end of the paper.

2) $(x, p) \equiv (x^i, p^i_\alpha)$, $p^i_\alpha \equiv \partial x^i / \partial u^\alpha$.

where

$$\Gamma^{*\gamma}_{\lambda k} \equiv \Gamma^{*\gamma}_{sk} p^s_\lambda, \quad ,k \equiv \frac{\partial}{\partial x^k}, \quad ;^\lambda_\gamma \equiv \frac{\partial}{\partial p^\gamma_\lambda},$$

and the symbols 'the small vertical bar' and a 'long solidus' denote the operators of covariant derivations with respect to x^k and p^γ_λ respectively.

As well known, in an $A^{(m)}_n$ the following relations hold good:

$$F_{|i} = 0, \quad p^\gamma_{\lambda|j} = 0, \quad g_{ij|k} = 0 \quad \text{and} \quad p^\alpha_{i|j} = 0.$$

Now, in an $A^{(m)}_n$, let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + \xi^i(x)\, d\tau, \tag{2.3}$$

where $d\tau$ is an infinitesimal constant and $\xi^i(x)$ is a contravariant vector field defined over the domain $\mathscr{R}$ of the space under consideration which is independent of the direction and is also at least of class C^2.

With respect to the transformation (2.3), T. Igarashi [6] has already developed the theory of Lie-derivatives in an $A^{(m)}_n$. Here we write down the Lie-derivatives of a scalar $f(x, p)$, vector $X^i(x, p)$ a mixed tensor $T^i_{jk}(x, p)$ and of the connection parameters $\Gamma^{*i}_{jk}(x, p)$ as below:

$$\pounds f = f_{|i}\xi^i + f;^\lambda_\gamma \xi^\gamma_{|i} p^i_\lambda,$$

$$\pounds X^i = X^i_{|j}\xi^j + X^i;^\lambda_\gamma \xi^\gamma_{|j} p^j_\lambda - X^j \xi^i_{|j}, \tag{2.4}$$

$$\pounds T^i_{jk} = T^i_{jk|l}\xi^l + T^i_{jk};^\lambda_\gamma \xi^\gamma_{|l} p^l_\lambda - T^l_{jk}\xi^i_{|l} + T^i_{lk}\xi^l_{|j} + T^i_{jl}\xi^l_{|k}, \tag{2.5}$$

$$\pounds \Gamma^{*i}_{jk} = \xi^i_{|j|k} + R^i_{jkl}\xi^l + \Gamma^{*i}_{jk};^\alpha_l \xi^l_{|h} p^h_\alpha, \tag{2.6}$$

where the symbol $\pounds$ denotes the operator of the Lie-differentiation process and $R^i_{jkl}(x, p)$ is the curvature tensor of the $A^{(m)}_n$ which is obtained as the coefficient of the vector $\xi^i(x)$ in the commutation formula

$$\xi^i_{|j|k} - \xi^i_{|k|j} = R^i_{hjk}\xi^h.$$

Further it is seen with ease that $\pounds p^i_\alpha = 0$.

Now, we endeavour ourselves in deriving some formulae which will frequently be used in the latter part of this paper. For the purpose in hand, let us first have the identities:

$$T^i_{jk|l};^\lambda_\gamma - T^i_{jk};^\lambda_{\gamma|l} = T^h_{jk}\Gamma^{*i}_{hl};^\lambda_\gamma - T^i_{hk}\Gamma^{*h}_{jl};^\lambda_\gamma - T^i_{jh}\Gamma^{*h}_{kl};^\lambda_\gamma - T^i_{jk};^\beta_s \Gamma^{*s}_{tl};^\lambda_\gamma p^t_\beta, \tag{2.7}$$

$$T^i_{jk|l|m} - T^i_{jk|m|l} = T^h_{jk}R^i_{hlm} - T^i_{hk}R^h_{jlm} - T^i_{jh}R^h_{klm} - T^i_{jk};^\lambda_\gamma R^\gamma_{\lambda lm}. \tag{2.8}$$

Using (2.5) and (2.7), on one hand, we obtain

$$(\pounds T^i_{jk});^\lambda_\gamma - \pounds(T^i_{jk};^\lambda_\gamma) = 0. \tag{2.9}$$

On the other hand, from (2.5) and (2.8), we can calculate that

$$\pounds(T^i_{jk|l}) - (\pounds T^i_{jk})_{|l} = T^h_{jk}\pounds\Gamma^{*i}_{hl} - T^i_{hk}\pounds\Gamma^{*h}_{jl} - T^i_{jh}\pounds\Gamma^{*h}_{kl} - T^i_{jk};^\lambda_\gamma(\pounds\Gamma^{*\gamma}_{sk})p^s_\lambda. \tag{2.10}$$

Next, we recall the formula (Om P. Singh [3])

$$(\pounds\Gamma^{*i}_{jk})_{|l} - (\pounds\Gamma^{*i}_{jl})_{|k} = \pounds R^i_{jkl} + \Gamma^{*i}_{jk};^\lambda_\gamma(\pounds\Gamma^{*\gamma}_{sl})p^s_\lambda - \Gamma^{*i}_{jl};^\lambda_\gamma(\pounds\Gamma^{*\gamma}_{sk})p^s_\lambda. \tag{2.11}$$

Employing the formula (2.10) for the normalized metric tensor [7] and using the property $g_{ij|k}=0$, we find that

$$(2.12)\qquad (\pounds\, g_{jk})_{|l} = g_{hk}\,\pounds\,\Gamma^{*}{}^{h}_{jl} + g_{jh}\,\pounds\,\Gamma^{*}{}^{h}_{kl} + g_{jk};{}^{\lambda}_{\gamma}(\pounds\,\Gamma^{*}{}^{\gamma}_{sl})p^{s}_{\lambda},$$

from which, on putting $C_{ij},{}^{\lambda}_{\gamma}+C_{ji},{}^{\lambda}_{\gamma} = g_{ij};{}^{\lambda}_{\gamma} \equiv \overset{\times}{C}_{ij},{}^{\lambda}_{\gamma}$, by straight forward calculation, we can have

$$(2.13)\qquad \begin{aligned} 2\,\pounds\,\Gamma^{*}{}^{i}_{jk} &= g^{ih}[(\pounds\, g_{hj})_{|k} + (\pounds\, g_{hk})_{|j} - (\pounds\, g_{jk})_{|h}] - \\ &- \overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma}(\pounds\,\Gamma^{*}{}^{\gamma}_{s\lambda})p^{s}_{k} - \overset{\times}{C}{}^{i}_{k},{}^{\lambda}_{\gamma}(\pounds\,\Gamma^{*}{}^{\gamma}_{sj})p^{s}_{\lambda} + g^{ih}\overset{\times}{C}_{jk},{}^{\lambda}_{\gamma}(\pounds\,\Gamma^{*}{}^{\gamma}_{sh})p^{s}_{\lambda}. \end{aligned}$$

Finally, using (2.2) and (2.4), after some calculation, we can derive the formula

$$(2.14)\qquad \pounds\,(X^{i}|^{\lambda}_{\gamma}) - (\pounds\, X^{i})|^{\lambda}_{\gamma} = X^{h}\,\pounds\, C^{i}_{h},{}^{\lambda}_{\gamma}.$$

Now we consider a set of r infinitesimal point transformations

$$\bar{x}^{i} = x^{i} + \xi^{i}_{a}(x)\,d\tau, \quad a, b, c, \ldots = 1, 2, \ldots, r.$$

If we denote the Lie-derivatives with respect to the vector ξ^{i}_{a} by $\underset{a}{\pounds}$, commutating the order of operators $\underset{a}{\pounds}$ and $\underset{b}{\pounds}$ on a scalar $f(x, p)$, we obtain

$$(\underset{b}{\pounds}\underset{a}{\pounds} - \underset{a}{\pounds}\underset{b}{\pounds})f = f_{|i}(\xi^{i}_{a|j}\xi^{j}_{b} - \xi^{j}_{a}\xi^{i}_{b|j}) + f;{}^{\alpha}_{h}(\xi^{h}_{a|j}\xi^{j}_{b} - \xi^{j}_{a}\xi^{h}_{b|j})_{|i}\,p^{i}_{\alpha},$$

which, on putting

$$\underset{b}{\pounds}\,\xi^{i}_{a} = \xi^{i}_{a|j}\xi^{j}_{b} - \xi^{j}_{a}\xi^{i}_{b|j},$$

can be written as

$$(2.15)\qquad (\underset{b}{\pounds}\underset{a}{\pounds})f \equiv (\underset{b}{\pounds}\underset{a}{\pounds} - \underset{a}{\pounds}\underset{b}{\pounds})f = f_{|i}(\underset{b}{\pounds}\,\xi^{i}_{a}) + f;{}^{\alpha}_{h}(\underset{b}{\pounds}\,\xi^{h}_{a})_{|i}\,p^{i}_{\alpha}.$$

Likewise, for the vector $U^{i}(x, p)$, tensor $T^{i}_{jk}(x, p)$ and components of connection parameters $\Gamma^{*}{}^{i}_{jk}(x, p)$, following relations can be derived easily

$$(2.16)\qquad (\underset{b}{\pounds}\underset{a}{\pounds})U^{i} = U^{i}_{|j}(\underset{b}{\pounds}\,\xi^{j}_{a}) + U^{i};{}^{\alpha}_{h}(\underset{b}{\pounds}\,\xi^{h}_{a})_{|j}\,p^{j}_{\alpha} - U^{j}(\underset{b}{\pounds}\,\xi^{i}_{a})_{|j},$$

$$(2.17)\qquad (\underset{b}{\pounds}\underset{a}{\pounds})\,T^{i}_{jk} = T^{i}_{jk|l}(\underset{b}{\pounds}\,\xi^{l}_{a}) + T^{i}_{jk};{}^{\alpha}_{h}(\underset{b}{\pounds}\,\xi^{h}_{a})_{|l}\,p^{l}_{\alpha} - T^{l}_{jk}(\underset{b}{\pounds}\,\xi^{i}_{a})_{|l} + T^{i}_{lk}(\underset{b}{\pounds}\,\xi^{l}_{a})_{|j} + T^{i}_{jl}(\underset{b}{\pounds}\,\xi^{l}_{a})_{|k},$$

$$(2.18)\qquad (\underset{b}{\pounds}\underset{a}{\pounds})\,\Gamma^{*}{}^{i}_{jk} = (\underset{b}{\pounds}\,\xi^{i}_{a})_{|j|k} + R^{i}_{jkl}(\underset{b}{\pounds}\,\xi^{l}_{a}) + \Gamma^{*}{}^{i}_{jk};{}^{\alpha}_{h}(\underset{b}{\pounds}\,\xi^{h}_{a})_{|l}\,p^{l}_{\alpha}.$$

Consequently, we can enunciate the

Theorem 2.1. *In an $A^{(m)}_{n}$ if we apply the operations $(\underset{b}{\pounds}\underset{a}{\pounds})$ to an arbitrary scalar $f(x, p)$, vector $U^{i}(x, p)$, tensor $T^{i}_{jk}(x, p)$ and the components of connection $\Gamma^{*}{}^{i}_{jk}(x, p)$, the results are also the Lie-derivatives of these quantities with respect to the vector $\underset{b}{\pounds}\,\xi^{i}_{a}$.*

Furthermore, if we consider an r-parameter group G_γ of transformations whose symbols are $\underset{a}{\pounds} f$ which are r-infinitesimal operators of an r-parameter group G_γ, we have

$$(\underset{b}{\pounds}\underset{a}{\pounds})f = C^c_{ba}\underset{c}{\pounds} f \quad \text{or} \quad \underset{b}{\pounds}\,\xi^i_a = C^c_{ba}\xi^i_c, \tag{2.19}$$

where C^c_{ba} said to be the structural constants of the group G_γ, are constants. Consequently, the relations (2.16), (2.17) and (2.18) are reduced to

$$(\underset{b}{\pounds}\underset{a}{\pounds})U^i = C^c_{ba}\underset{c}{\pounds} U^i, \quad (\underset{b}{\pounds}\underset{a}{\pounds})T^i_{jk} = C^c_{ba}\underset{c}{\pounds} T^i_{jk} \quad \text{and} \quad (\underset{b}{\pounds}\underset{a}{\pounds})\Gamma^{*}{}^i_{jk} = C^c_{ba}\underset{c}{\pounds}\Gamma^{*}{}^i_{jk},$$

respectively. Hence, it follows:

Theorem 2.2. *When ξ^i_a are r-vectors of an r-parameter group G_γ of infinitesimal transformations in an $A^{(m)}_n$, the similar type of relations as* (2.19) *hold good for an arbitrary scalar, vector, tensor and the components of connection $\Gamma^{*}{}^i_{jk}$ in an areal space of submetric class.*

§ 3. Group of homothetic transformations

In our previous paper [3], we have defined the homothetic transformation by means of a relation. Here we start our discussion by giving the geometrical interpretation of a homothetic transformation to show its physical significance in our space and then we shall prove the validity of our relation (3.2) in [3].

In an $A^{(m)}_n$, let us consider a point transformation

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n), \quad \text{provided that} \quad \left|\frac{\partial \bar{x}^i}{\partial x^j}\right| \neq 0, \tag{3.1}$$

which establishes a one to one correspondence between the points of a region R and those of some other region $\bar{R}$. Then we can assume that, under this transformation, p^i_α is transformed to $\bar{p}^i_\alpha \equiv p^j_\alpha \dfrac{\partial \bar{x}^i}{\partial x^j}$, an original point x^i in R is carried to a displaced point $\bar{x}^i$ in $\bar{R}$, and a point $x^i + dx^i$ in R to a point $\bar{x}^i + d\bar{x}^i$ in $\bar{R}$. If the distance between two sufficiently near points x^i and $x^i + dx^i$ is transformed to that between the corresponding two sufficiently near points $\bar{x}^i$ and $\bar{x}^i + d\bar{x}^i$ in a constant ratio under the transformation (3.1), we call this transformation a homothetic transformation in the space under consideration. In case the transformation (3.1) is a homothetic one, a necessary condition what we have, is that two normalized metric tensors $\bar{\bar{g}}_{ij}(x, p)$ and $g_{ij}(x, p)$ bear a constant ratio i. e.,

$$\bar{\bar{g}}_{ij}(x, p) = k g_{ij}(x, p), \tag{3.2}$$

where $\bar{\bar{g}}_{ij}(x, p)$ is the deformed counter part [4] of the normalized metric tensor $g_{ij}(x, p)$ under the transformation (3.1) and k is a positive constant.

We now consider the case in which the point transformation (3.1) becomes an infinitesimal one

$$\bar{x}^i = x^i + \xi^i(x)\, d\tau. \tag{3.3}$$

In such a case, if the transformation (3.3) is an infinitesimal homothetic one, then from (3.2), we must have

$$(3.4)\qquad \pounds\, g_{ij} = 2Cg_{ij},$$

where C is a constant. Conversely, if (3.4) holds good, then applying the definition of the deformed quantity of the normalized metric tensor $g_{ij}(x, p)$ [4], we can have $\bar{\bar{g}}_{ij} = g_{ij} + \pounds\, g_{ij}\, d\tau$ from which we get $\bar{\bar{g}}_{ij}(x, p) = kg_{ij}(x, p)$, where $k(c, d\tau)$ is a positive constant determined by c and $d\tau$. Thus, we have.

Theorem 3.1. *In order that an infinitesimal point transformation* (3.3) *in an* $A_n^{(m)}$ *be a homothetic one, it is necessary and sufficient that the relation* (3.4) *holds good.*

In relation (3.2), if $k=1$, the transformation (3.1) is a motion, and likewise, when C in (3.4) is zero, the infinitesimal transformation (3.3) is also a motion [6]. Therefore, we shall consider only the case $C \neq 0$ throughout this paper and we call a homothetic transformation for which $C \neq 0$ (or which is not a motion) a proper infinitesimal homothetic transformation.

Further, we suppose that each of r linearly independent infinitesimal operators $\underset{a}{\pounds} f$ defines a one parameter group of homothetic transformations. If any operator $\pounds f$ is a linear combination of $\underset{a}{\pounds} f$ and constant coefficients, then the set of these operators $\pounds f$ is said to be complete. Now the following theorem holds:

Theorem 3.2 *In an* $A_n^{(m)}$, *if the infinitesimal operators* $\underset{a}{\pounds} f$ *are r-generators of a complete set of one-parameter groups of homothetic transformations, then, they are also generators of an r-parameter group of homothetic transformations.*

If the symbols $\underset{a}{\pounds} f$ are generators of r one-parameter groups of transformations, we have $(\underset{b}{\pounds}\underset{a}{\pounds})f = \underset{ba}{\pounds} f$ or $(\underset{b}{\pounds}\underset{a}{\pounds})g_{ij} = \underset{ba}{\pounds}\, g_{ij}$, then the symbols $\underset{ba}{\pounds} f$ are generators of one parameter groups whose vector fields are $\underset{b}{\pounds}\, \xi_a^i$. On the other hand, when the symbols $\underset{a}{\pounds} f$ are generators of r one-parameter group of homothetic transformations, then by virtue of (3.4), we have $(\underset{b}{\pounds}\underset{a}{\pounds})\, g_{ij} = 0$. Consequently, we get $\underset{ba}{\pounds}\, g_{ij} = 0$. Furthermore, in case of $\underset{a}{\pounds} f$ being r generators of an r-parameter groups of transformations, from theorem 2.2, it is evident that $(\underset{b}{\pounds}\underset{a}{\pounds})g_{ij} = \underset{ba}{\pounds}\, g_{ij} = C_{ba}^c \underset{c}{\pounds}\, g_{ij}$, where C_{ba}^c are the structural constants of the group, but for an r-parameter group of homothetic transformations, we also have $(\underset{b}{\pounds}\underset{a}{\pounds})f = 0$ and $\underset{a}{\pounds}\, g_{ij} = 2C_a g_{ij}$. Therefore, we at once find that $C_{ba}^c C_c = 0$. Hence, summarizing, the results from above, it follows:

Theorem 3.3. *In an* $A_n^{(m)}$ *if the symbols* $\underset{a}{\pounds} f$ *are generators of r one-parameter groups of homothetic transformations, then, the symbols* $\underset{ba}{\pounds} f$ *are those of a one-parameter group of areal motions in the same space.*

Theorem 3.4. *In an* $A_n^{(m)}$, *if the symbols* $\underset{a}{\pounds} f$ *are r generators of an r-parameter group of homothetic transformations with homothetic constants* C_a, *then we have a relations* $C_{ba}^c C_c = 0$ *between the structural constants* C_{ba}^c *and the homothetic constants* C_c.

Finally, we note that if $\underset{a}{\pounds} f$ are generators of an r-parameter group G_γ of transformations, the totality of $\underset{ba}{\pounds} f$ determines in general a group of transformations known as derived group which is either G_γ itself or a normal subgroup of G_γ. Hence, from this and theorem 3.6, we state

Theorem 3.5. *The derived group of an r-parameter group of homothetic transformations in an $A_n^{(m)}$ is a group of areal motions in the same space.*

§ 4. Integrability conditions of $\pounds g_{ij} = 2Cg_{ij}$

Applying (2.5) to g_{ij}, in virtue of $g_{ij|k} = 0$ and on putting $g_{ij};^\lambda_\gamma = \overset{\times}{C}_{ij};^\lambda_\gamma$, if we introduce (3.4), we have

$$\pounds g_{ij} = g_{hj}\xi^h_{|i} + g_{ih}\xi^h_{|j} + \overset{\times}{C}_{ij},^\lambda_\gamma \xi^\gamma_{|h} p^h_\lambda = 2Cg_{ij}. \tag{4.1}$$

Next, on making use of formula (2.10) for g_{ij}, employing (3.4) again, and knowing that $g_{ij|k} = 0$, it is obviously seen that $\pounds \Gamma^{*h}_{ij} = 0$. Thus, from (2.6), we get

$$\pounds \Gamma^{*i}_{jk} = \xi^i_{|j|k} + R^i_{jkl}\xi^l + \Gamma^{*i}_{jk};^\alpha_\gamma \xi^l_{|h} p^h_\alpha = 0. \tag{4.2}$$

We shall now endeavour ourselves in considering the conditions that, equation (4.1) admits a set of solutions $\xi^i(x)$ and C, $\xi^i(x)$ being functions of x^i only and C a constant. Clearly, any solution $\xi^i(x)$ of (4.1) always satisfies the equations (4.2). Therefore, our conditions are the integrability conditions of the differential equations. The equations (4.2) with the conditions (4.1) may be written as a mixed system of partial differential equations

$$\xi^i_{|j} = \xi^i_j, \quad \xi^i;^\lambda_\gamma = 0, \quad \xi^i_j;^\lambda_\gamma = \xi^l \Gamma^{*i}_{lj};^\lambda_\gamma, \quad C_{|i} = 0, \quad C;^\alpha_j = 0,$$
$$\xi^i_{|j|k} = -R^i_{jkl}\xi^l - \Gamma^{*i}_{jk};^\lambda_\gamma \xi^\gamma_l p^l_\lambda = 0,$$

together with the condition

$$g_{hj}\xi^h_i + g_{ih}\xi^h_j + \overset{\times}{C}_{ij},^\lambda_\gamma \xi^\gamma_h p^h_\lambda = 2Cg_{ij}.$$

When an infinitesimal point transformation (3.3) is a homothetic one, from (3.4), by means of (2.9) and because of $g_{ij};^\lambda_\gamma = \overset{\times}{C}_{ij},^\lambda_\gamma$, first we can have

$$\text{a) } \pounds g^{ij} = -2Cg^{ij}, \qquad \text{b) } \pounds \overset{\times}{C}_{ij},^\lambda_\gamma = 2C\overset{\times}{C}_{ij},^\lambda_\gamma. \tag{4.3}$$

Secondly, from the above relations and in virtue of $g^{ih}\overset{\times}{C}_{hj},^\lambda_\gamma = \overset{\times}{C}{}^i_j,^\lambda_\gamma$, we can deduce that

$$\pounds \overset{\times}{C}{}^i_j,^\lambda_\gamma = 0. \tag{4.4}$$

Now, taking aid of (4.2), from (2.11) and (2.9), we can get with case

$$\pounds R^i_{jkl} = 0, \tag{4.5}$$

$$\pounds \Gamma^{*i}_{jk};^\lambda_\gamma = 0. \tag{4.6}$$

It is well known that in an $A_n^{(m)}$ the connection $\Gamma^{*}{}^{i}_{jk}(x, p)$ of KAWAGUCHI and TANDAI [7] satisfies the famous identity of Ricci that is, the associated covariant derivative of normalized matric tensor g_{ij} is zero, so we have

$$g_{ij|k} = g_{ij},_{k} - g_{ij};{}^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{\lambda k} - g_{hj}\Gamma^{*}{}^{h}_{ik} - g_{ih}\Gamma^{*}{}^{h}_{jk} = 0.$$

From this, we can easily obtain

$$\Gamma^{*}{}^{i}_{jk} = \{{}^{i}_{jk}\} - \tfrac{1}{2}\{\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{\lambda k} + \overset{\times}{C}{}^{i}_{k},{}^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{\lambda j} - g^{ih}\overset{\times}{C}_{jk},{}^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{\lambda h}\}.$$

Differentiating these equation with respect to p^{s}_{μ}, we have

$$\Gamma^{*}{}^{i}_{jk};{}^{\mu}_{s} = \tfrac{1}{2}[\overset{\times}{C}{}^{i}_{j},{}^{\mu}_{s|k} + \overset{\times}{C}{}^{i}_{k},{}^{\mu}_{s|j} - \overset{\times}{C}_{jk},{}^{\mu}_{s|h}g^{hi} -$$

(4.7)

$$-(\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{tk};{}^{\mu}_{s} + \overset{\times}{C}{}^{i}_{k},{}^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{tj};{}^{\mu}_{s} - \overset{\times}{C}_{jk},{}^{\lambda}_{\gamma}\Gamma^{*}{}^{\gamma}_{th};{}^{\mu}_{s}g^{ih})p^{t}_{\mu}].$$

Further, making the use of formula (2.10), (4.2), (4.3) and (4.4), it can be verified that

$$\pounds\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma|k} = 0, \quad \pounds\overset{\times}{C}_{ij},{}^{\lambda}_{\gamma|k} = 2C\overset{\times}{C}_{ij},{}^{\lambda}_{\gamma|k}. \tag{4.8}$$

Operating $\pounds$ on both sides of (4.7), using (4.3), (4.4) and (4.8), we obtain

$$\pounds\Gamma^{i}_{jk};{}^{\mu}_{s} = -\tfrac{1}{2}(\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma}\pounds\Gamma^{*}{}^{\gamma}_{tk};{}^{\mu}_{s} + \overset{\times}{C}{}^{i}_{k},{}^{\lambda}_{\gamma}\pounds\Gamma^{*}{}^{\gamma}_{tj};{}^{\mu}_{s} - \overset{\times}{C}_{jk};{}^{\lambda}_{\gamma}(\pounds\Gamma^{*}{}^{\gamma}_{th};{}^{\mu}_{s})g^{ih})p^{t}_{\mu}. \tag{4.9}$$

Finaly, on transvecting (4.9) with p^{j}_{α}, we can find that

$$W^{il\lambda}_{\gamma k\alpha}(\pounds\Gamma^{*}{}^{\gamma}_{tl};{}^{\mu}_{s})p^{t}_{\lambda} = 0. \tag{4.10}$$

Where

$$W^{il\lambda}_{\gamma k\alpha} = \delta^{i}_{\gamma}\delta^{l}_{k}\delta^{\lambda}_{\alpha} + \tfrac{1}{2}(\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma}\delta^{l}_{k} + \overset{\times}{C}{}^{i}_{k},{}^{\lambda}_{\gamma}\delta^{l}_{j} - g^{il}\overset{\times}{C}_{jk},{}^{\lambda}_{\gamma})p^{j}_{\alpha}.$$

Since we notice that the mn^2-rowed determinant W constructed of $W^{il\lambda}_{\gamma k\alpha}$ with respect to the system of indices $(ik\alpha)$ and $(\gamma l\lambda)$ doesn't vanish identically in the considered domain. Therefore, from (4.10), we get $(\pounds\Gamma^{*}{}^{\gamma}_{tl};{}^{\mu}_{s})p^{t}_{\mu}=0$. Substituting this result in (4.9), we obtain $\pounds\Gamma^{*}{}^{i}_{jk};{}^{\mu}_{s}=0$. Consequently, we remark that conditions (4.5) and (4.6) due to (4.2) are not needed to be considered both at a time, for the latter is a consequence of (4.4). Therefore, we shall only consider the conditions (4.4), (4.5) and their successive covariant derivatives.

Now we consider first the condition (4.4). Making use of (2.9) for $\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma}$ and emplying (4.4), we see that $\pounds\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma};{}^{\mu}_{s}=0$. Again, following the same procedure for the repeated partial derivatives of $\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma}$ with respect to p^{γ}_{λ}, we can get

$$\pounds\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma};{}^{\mu_1}_{s_1};{}^{\mu_2}_{s_2};\dots;{}^{\mu_p}_{s_p} = 0, \quad p = 1, 2, 3, \dots.$$

Since the left hand side of this equation is throughout homogeneous with respect to p^{γ}_{λ}, so because of the reason $\pounds p^{\gamma}_{\lambda}=0$, we note that this equation retains all the preceding one. Therefore, the above equation can be written as

$$\pounds\overset{\times}{C}{}^{i}_{j},{}^{\lambda}_{\gamma};{}^{\mu_1}_{s_1};{}^{\mu_2}_{s_2};\dots;{}^{\mu_p}_{s_p} = 0.$$

Of course, by means of the formula (2.7), it can be seen that the conditions obtained from (4.4) by applying first the covariant differentiation and next the partial differentiation and those obtained vice versa are equivalent. Thus, we consider those conditions which are obtained from (4.4) by applying successively only the partial differentiations with respect to p^{γ}_{λ} i.e.,

$$\pounds \overset{\times}{C}{}^{i}_{j},{}^{;\lambda}_{;\gamma}{}^{;\mu_1}_{;s_1}{}^{;\mu_2}_{;s_2}{}^{;\ldots;\mu_p}_{;\ldots;s_p} = 0.$$

Now, following the validity of our above statement, by repeated covariant differentiation of this latter equation, we get

$$(\pounds \overset{\times}{C}{}^{i}_{j},{}^{;\lambda}_{;\gamma}{}^{;\mu_1}_{;s_1}{}^{;\mu_2}_{;s_2}{}^{;\ldots;\mu_p}_{;\ldots;s_p})_{|k_1|k_2|\ldots|k_t} = 0, \quad t = 1, 2, 3, \ldots,$$

which are the conditions so obtained from (4.4) by successive partial and covariant differentiations.

Secondly, we consider the condition (4.6). The implication of the formula (2.9) for the tensor R^{i}_{jkl} together with the condition (4.5) yields the result $\pounds R^{i}_{jkl}{}^{;\lambda}_{;\gamma} = (\pounds R^{i}_{jkl}){}^{;\lambda}_{;\gamma} = 0$. Furthermore, on one hand, applying the formula (2.10) to $\Gamma^{*i}_{jk}{}^{;\lambda}_{;\gamma}$, we have

$$\pounds \Gamma^{*i}_{jk}{}^{;\lambda}_{;\gamma|l} = (\pounds \Gamma^{*i}_{jk}{}^{;\lambda}_{;\gamma})_{|l} + \Gamma^{*h}_{jk}{}^{;\lambda}_{;\gamma} \pounds \Gamma^{*i}_{hl} - \Gamma^{*i}_{hk}{}^{;\lambda}_{;\gamma} \pounds \Gamma^{*h}_{jl} - \Gamma^{*i}_{jh}{}^{;\lambda}_{;\gamma} \pounds \Gamma^{*h}_{kl} -$$
$$- \Gamma^{*i}_{jk}{}^{;\lambda}_{;h} \pounds \Gamma^{*h}_{\gamma l} - (\Gamma^{*i}_{jk}{}^{;\lambda}_{;\gamma}{}^{;\mu}_{;s})(\pounds \Gamma^{*s}_{hl})\, p^{h}_{\mu}.$$

and, on the other hand, if we apply the operator $\pounds$ to the relation (Om P. Singh [3])

$$R^{h}_{ijk}{}^{;\alpha}_{;l} = (\Gamma^{*h}_{ij}{}^{;\alpha}_{;l})_{|k} - (\Gamma^{*h}_{ik}{}^{;\alpha}_{;l})_{|j} - \Gamma^{*h}_{ij}{}^{;\beta}_{;m} \Gamma^{*m}_{\beta k}{}^{;\alpha}_{;l} + \Gamma^{*h}_{ik}{}^{;\beta}_{;m} \Gamma^{*m}_{\beta j}{}^{;\alpha}_{;l}.$$

We obtain

$$\pounds R^{h}_{ijk}{}^{;\alpha}_{;l} = \pounds \Gamma^{*h}_{ij}{}^{;\alpha}_{;l|k} - \pounds \Gamma^{*h}_{ik}{}^{;\alpha}_{;l|j} - (\pounds \Gamma^{*h}_{ij}{}^{;\beta}_{;m}) \Gamma^{*m}_{\beta k}{}^{;\alpha}_{;l} - \Gamma^{*h}_{ij}{}^{;\beta}_{;m} (\pounds \Gamma^{*m}_{\beta k}{}^{;\alpha}_{;l}) +$$
$$+ (\pounds \Gamma^{*h}_{ik}{}^{;\beta}_{;m}) \Gamma^{*m}_{\beta j}{}^{;\alpha}_{;l} - \Gamma^{*h}_{ik}{}^{;\beta}_{;m} (\pounds \Gamma^{*m}_{\beta j}{}^{;\alpha}_{;l}).$$

Substituting the former into the latter one and employing (4.2), (4.6) and its covariant derivatives, we get $\pounds R^{i}_{jkl}{}^{;\lambda}_{;\gamma} = 0$, from which it implies that the equations obtained from (4.5) by partial differentiation with respect to p^{γ}_{λ} doesn't give further new conditions.

Again, from (4.5), by using the formula (2.10) and (4.2), we obtain $(\pounds R^{h}_{ijk})_{|l_1} = 0$. Here we can also see that the conditions obtained from (4.5) by first applying the partial differentiation w.r.t. p^{γ}_{λ} and then covariant differentiation, and those obtained vice versa are equivalent. Thus, we can also realize with ease that the partial differentiation of the above equations with respect to p^{γ}_{λ} doesn't give new conditions.

From the latest equation, we may find that $(\pounds R^{h}_{ijk})_{|l_1|l_2} = 0$, but in a likewise manner, we can again show that the partial derivative of this equation w.r.t. p^{γ}_{λ} doesn't give further new conditions.

Repeating the similar process, we get at last

$$(\pounds R^{h}_{ijk})_{|l_1|l_2|\ldots|l_q} = 0.$$

From the above discussion, we can give the

Theorem 4.1. *In order that* (4.1) *have a set of solutions* $\xi^i(x)$ *which are only point functions, and* C *which is a constant, it is necessary and sufficient that the equations*

$$g_{hj}\xi^h_{|i}+g_{ih}\xi^h_{|j}+\overset{\times}{C}_{ij},{}^{\lambda}_{\gamma}\xi^{\gamma}_{|h}p^h_{\lambda}=2Cg_{ij}, \tag{4.11}$$

and

$$\begin{cases}(\pounds\overset{\times}{C}{}^i_j,{}^{\lambda}_{\gamma};{}^{\mu_1}_{s_1};{}^{\mu_2}_{s_2};\dots;{}^{\mu_p}_{s_p})_{|k_1|k_2|\dots|k_t}=0,\\ (\pounds R^h_{i\,jk})_{|l_1|l_2|\dots|l_q}=0,\quad t,q=0,1,2,\dots\end{cases} \tag{4.12}$$

be algebraially consistent in ξ^i, $\xi^i_{|j}$ *and* C.

Clearly, the solutions of (4.11) and (4.12) are linearly homogenous in ξ^i, $\xi^i_{|j}$ and C, and there are $\frac{1}{2}N(N+1)$ linearly independent equations in (4.11). Therefore the maximum number of linearly independent solutions ξ^i, $\xi^i_{|j}$ and C which satisfy (4.11) and (4.12), is $\frac{1}{2}N(N+1)+1$. Consequently, it follows the

Theorem 4.2. *In order that the space* $A_n^{(m)}$ *admit* $r\left(\leqq\frac{1}{2}N(N+1)+1\right)$ *linearly independent proper infinitesimal homothetic transformations, it is necessary and sufficient that* (4.11) *and* (4.12) *involve* $N(N+1)+1-r$ *linearly independent equations and the others be consequences of them.*

In case of r linearly independent infinitesimal homothetic transformations, if we denote them by $\bar{x}^i=x^i+\overset{*}{\xi}{}^i_a(x)d\tau$, any $\xi^i_a(x)$ which satisfies (4.1), is a linear combination of $\overset{*}{\xi}{}^i_a(x)$ with constant coefficients. But in virtue of theorem 2.1 and theorem 3.1, as a fact is obvoius that $\underset{b}{\pounds}\,\xi^i_a$ also satisfy (4.1). Consequently, $\underset{b}{\pounds}\,\xi^i_a$ should necessarily be the linear combination of $\overset{*}{\xi}{}^i_a(x)$ with constant coefficients C. Thus, from theorem 3.5 and theorem 4.2. it follows:

Theorem 4. 3. *In order that the the space* $A_n^{(m)}$ *admit an r-parameter group of poper infinitesimal homothetic transformations, it is necessary and sufficient that* (4.11) *and* (4.12) *involve* $N(N+1)+1-r$ *linearly independent equations and the others be consequences of them.*

At last, we devote ourselves to determine the condition that the space $A_n^{(m)}$ admits a group of proper infinitesimal homothetic transformations of maximum order $\frac{1}{2}N(N+1)+1$. In order to be the case, a necessary and sufficient condition what we have, is that the quations $\pounds\overset{\times}{C}{}^i_j,{}^{\lambda}_{\gamma}=0$ and $\pounds R^i_{jkl}=0$ should identicially be satisfied for any ξ^i, $\xi^i_{|j}$ and C such that

$$\pounds g_{ij}=\xi_{i|j}+\xi_{j|i}+\overset{\times}{C}_{ij},{}^{\lambda}_{\gamma}\xi^{\gamma}_{|h}p^h_{\lambda}=2Cg_{ij}, \tag{4.13}$$

where

$$\xi_{i|j}=g_{ih}\xi^h_{|j}.$$

When the space $A_n^{(m)}$ admits a group of proper infinitesimal homothetic transformations of maximum order, from (4.4) and (4.5), we have

$$\pounds\overset{\times}{C}{}^i_j,{}^{\lambda}_{\gamma}=\overset{\times}{C}{}^i_j,{}^{\lambda}_{\gamma|h}\xi^h+\overset{\times}{C}{}^i_j,{}^{\lambda}_{\gamma};{}^{\mu}_{s}\xi^s_{|h}p^h_{\mu}-\overset{\times}{C}{}^h_j,{}^{\lambda}_{\gamma}\xi^i_{|h}+\overset{\times}{C}{}^i_h,{}^{\lambda}_{\gamma}\xi^h_{|j}+\overset{\times}{C}{}^i_j,{}^{\lambda}_{h}\xi^h_{|\gamma}=0, \tag{4.14}$$

$$\pounds R^i_{jkl}=R^i_{jkl|h}\xi^h+R^i_{jkl};{}^{\mu}_{s}\xi^s_{|h}p^h_{\mu}-R^h_{jkl}\xi^i_{|h}+R^i_{hkl}\xi^h_{|j}+R^i_{jhl}\xi^h_{|k}+R^i_{jkh}\xi^h_{|l}=0. \tag{4.15}$$

In (4.14) and (4.15), the coefficients of ξ^h and $\xi^s_{|h}$ are

$$\overset{\times}{C}{}^i_{j},{}^{\lambda}_{\gamma|h};\ (\overset{\times}{C}{}^i_{j},{}^{\lambda}_{\gamma};{}^{\mu}_{s}p^h_\mu - \overset{\times}{C}{}^h_{j},{}^{\lambda}_{\gamma}\delta^i_s + \overset{\times}{C}{}^i_{s},{}^{\lambda}_{\gamma}\delta^h_j + \overset{\times}{C}{}^i_{j},{}^{\lambda}_{s}\delta^h_\gamma),$$

and

$$R^i_{jkl|h};\ (R^i_{jkl};{}^{\mu}_{s}p^h_\mu - R^h_{jkl}\delta^i_s + R^i_{skl}\delta^h_j + R^i_{jsl}\delta^h_k + R^i_{jks}\delta^h_l),$$

respectively.

Since the vector field $\xi^i(x)$ is arbitrary and independent of direction in itself, therefore it is immediately obvious that

$$\overset{\times}{C}{}^i_{j},{}^{\lambda}_{\gamma|h} = 0;\ \overset{\times}{C}{}^i_{j},{}^{\lambda}_{\gamma};{}^{\mu}_{s}p^h_\mu - \overset{\times}{C}{}^h_{j},{}^{\lambda}_{\gamma}\delta^i_s + \overset{\times}{C}{}^i_{s},{}^{\lambda}_{\gamma}\delta^h_j + \overset{\times}{C}{}^i_{j},{}^{\lambda}_{s}\delta^h_\gamma = 0, \tag{4.16}$$

$$R^i_{jkl|h} = 0;\ R^i_{jkl};{}^{\mu}_{s}p^h_\mu - R^h_{jkl}\delta^i_s + R^i_{skl}\delta^h_j + R^i_{jsl}\delta^h_k + R^i_{jks}\delta^h_l = 0. \tag{4.17}$$

Hence, in the case under consideration, for (4.13) holds good, from the second equations of (4.16) and (4.17), we must have

$$\overset{\times}{C}{}^i_{j},{}^{\lambda}_{\gamma} = 0 \quad \text{and} \quad R^i_{jkl} = 0.$$

But, it we consider the first equation of (4.16) and this last equation, then in virtue of theorem 6.2 of GAMA [5], we see that they constitute the conditions for the space $A_n^{(m)}$ to be *Minkowskian*. Moreover, the equation $\overset{\times}{C}{}^i_{j},{}^{\lambda}_{\gamma}=0$ alone shows that the tensor g_{ij} does not depend upon p^γ_λ i.e., the space $A_n^{(m)}$ is a Riemannian space, and the equation $R^i_{jkl}=0$ shows that the space $A_n^{(m)}$ is locally Euclidean.

Conversely, if $\overset{\times}{C}{}^i_{j},{}^{\lambda}_{\gamma}=0$ and $R^i_{jkl}=0$, then the second of equations (4.16) and (4.17) hold, an consequently, (4.12) is also valid. Hence the converse is also true, that is, the space $A_n^{(m)}$ admits a group of proper infinitesimal homothetic transformations of maximum order. Thus, we have

Theorem 4.4. *In order that the space $A_n^{(m)}$ admits a group of proper infinitesimal homothetic transformations of maximum order, it is necessary and sufficient that the space be locally Euclidean.*

We now recall that if the curvature tensor R^i_{jkl} of an $A_n^{(m)}$ $(n>2)$ satisfies the relation

$$R^i_{jkl} = K(\delta^i_l g_{jk} - \delta^i_k g_{jl}), \tag{4.18}$$

where K said to be the Riemannian curvature of our space, is a constant, then the space is a Riemannian space with constant curvature [8].

Operating $\underset{a}{£}$ on both the sides of (4.18), introducing (3.4) and then in virtue of (4.5), we have

$$\underset{a}{£}\, R^h_{jkh} = (n-1)KC_a g_{jk} = 0.$$

from which for $n>2$, and $C_a \neq 0$, we get $K=0$. Substituting this in (4.18), we notice that $R^i_{jkl}=0$. Thus our space is locally Euclidean. Hence, we can state the

Theorem 4.5. *In order that the space $A_n^{(m)}$ admits a group of proper infinitesimal homothetic transformations of maximum order, it is necessary and sufficient that the space be a Riemannian space with constant curvature.*

Theorem 4.6. *If the space $A_n^{(m)}$ of constant curvature admits a group of proper infinitesimal homothetic transformations, the space must be locally Euclidean.*

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