Representation of the solutions of the linear iterative equation by means of series

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The purpose of the present paper is to establish the general form of the solutions of the iterative equation:

$$f[\alpha(x)] = T[x, f(x)], \tag{1}$$

where f is — the unknown funtion, and T and α are given funtions verifying the conditions:

(i) α is a bijective application defined on a certain set X with values in the same set.

(ii) $T(x, y)$ is defined on the set $X \times Y$ — where Y is a Banach space. The range of T lies in Y. In addition, for any fixed $x \in X$, the function $T(x, y)$ is a linear (and bounded) isomorphism of Y on to Y.

Subsequently we shall denote by $T_x(y)$ the section through x of the function $T(x, y)$. Having in view the hypotheses made, it is meaningful to consider the following system of functions:

$$T_x^{(\nu)} = \begin{cases} T_x^{-1} \circ T_{\alpha(x)}^{-1} \circ \cdots \circ T_{\alpha^{\nu-1}(x)}^{-1} & \text{for} \quad \nu > 0 \\ E \text{ (the identical operator)} & \text{for} \quad \nu = 0 \\ T_{\alpha^{-1}(x)} \circ T_{\alpha^{-2}(x)} \circ \cdots \circ T_{\alpha^{\nu}(x)} & \text{for} \quad \nu < 0 \end{cases} \tag{2}$$

for any $x \in X$ and $\nu \in I$ (I — the set of the integers). Now we shall construct by means of the family defined by (2) a certain type of series, called iterative series. Let $u: X \to Y$ be an arbitrary function and let us consider the sums:

$$\sigma_n^+(u, x) = \sum_{k=0}^{n} T_x^{(k)}\big[u\big(\alpha^k(x)\big)\big], \quad \sigma_n^-(u, x) = \sum_{k=-1}^{-n} T_x^{(k)}\big[u\big(\alpha^k(x)\big)\big].$$

By means of these sums we have defined a pair of series.

Definition 1. The ordered pair of series $\big(\sigma_n^+(u, x), \sigma_n^-(u, x)\big)$, designated by the symbol $\sum_{k=-\infty}^{+\infty} T_x^{(k)}\big[u\big(\alpha^k(x)\big)\big]$ is called an iterative series attached to the function $u: X \to Y$.

Definition 2. An iterative series is called convergent, when both $\sigma_n^+(u, x)$ and $\sigma_n^-(u, x)$ are convergent; its sum is by definition $\sigma(u, x) = \lim_{n\to\infty} \sigma_n^+(u, x) + \lim_{m\to\infty} \sigma_m^-(u, x)$.

Remarks:

1. Further on, the sum of a series as well the series itself will be denoted by $\sigma(u, x)$ or $\sum_{k=-\infty}^{\infty} T_x^{(k)}[u(\alpha^k(x))]$.

2. If the iterative series $\sigma(u, x)$ is convergent, then:

(a) $\lim_{k\to\pm\infty} T_x^{(k)}[u(\alpha^k(x))] = \Theta$ (Θ = the null element from Y),

(aa) $\sigma(u, x) = \lim_{n\to\infty} \sum_{k=-n}^{n} T_x^{(k)}[u(\alpha^k(x))]$.

Theorem 1. *The set D of the points in which a given iterative series converges, is saturated (in the sense given in* [1]*).*

PROOF. Let $\sigma(u, x)$ be an iterative series, convergent in a point $x_0 \in X$. Let us point out that from $x_0 \in D$ there follows $\alpha(x_0), \alpha^{-1}(x_0) \in D$. To see this let us consider the expression:

$$\sigma_n^+(u, \alpha(x_0)) = \sum_{k=0}^{n} T_{\alpha(x_0)}^{(k)}[u(\alpha^k(\alpha(x_0)))] = \sum_{k=0}^{n} T_{\alpha(x_0)}^{(k)}[u(\alpha^{k+1}(x_0))].$$

But we have

$$T_{\alpha(x_0)}^{(k)} = T_{\alpha(x_0)}^{-1} \circ T_{\alpha^2(x_0)}^{-1} \circ \cdots \circ T_{\alpha^k(x_0)}^{-1} = T_{x_0} \circ T_{x_0}^{-1} \circ T_{\alpha(x_0)}^{-1} \circ \cdots \circ T_{\alpha^k(x_0)}^{-1} = T_{x_0} \circ T_{x_0}^{(k+1)},$$

thus:

$$\sigma_n^+(u, \alpha(x_0)) = \sum_{k=0}^{n} T_{x_0}[T_{x_0}^{(k+1)}(u(\alpha^{k+1}(x_0)))] = T_{x_0}\left[\sum_{k=0}^{n} T_{x_0}^{(k+1)}(u(\alpha^{k+1}(x_0)))\right]$$

and
$$\sigma_{n+p}^+(u, \alpha(x_0)) = T_{x_0}\left[\sum_{k=0}^{n+p} T_{x_0}^{(k+1)}(u(\alpha^{k+1}(x_0)))\right], \text{ i.e.,}$$

$$\left\|\sigma_{n+p}^+(u, \alpha(x_0)) - \sigma_n^+(u, \alpha(x_0))\right\| = \left\|T_{x_0}\left[\sum_{k=n+1}^{n+p} T_{x_0}^{(k+1)}(u(\alpha^{k+1}(x_0)))\right]\right\| =$$

$$= \left\|T_{x_0}\left[\sum_{i=n+2}^{n+p+1} T_{x_0}^{(i)}(u(\alpha^i(x_0)))\right]\right\| \leqq \|T_{x_0}\| \cdot \left\|\sum_{i=n+2}^{n+p+1} T_{x_0}^{(i)}(u(\alpha^i(x_0)))\right\|,$$

from this we infer that $\{\sigma_n^+(u, \alpha(x_0))\}_{n\in N}$ is a Cauchy-sequence, that is, convergent. Now put:

$$\sigma_n^-(u, x_0) = \sum_{k=-1}^{-n} T_{\alpha(x_0)}^{(k)}[u(\alpha^k(x_0))].$$

Since $k < 0$, we have:

$$T_{\alpha(x_0)}^{(k)} = T_{x_0} \circ T_{\alpha^{-1}(x_0)} \circ \cdots \circ T_{\alpha^k(x_0)} = T_{x_0} \circ T_{x_0}^{(k+1)}.$$

Further on, the reasoning is the same as in the previous stage. Therefore we have to proved that $\alpha^{-1}(x_0) \in D$, too.

We have:

$$\sigma_n^+(u, \alpha^{-1}(x_0)) = \sum_{k=0}^{n} T^{(k)}_{\alpha^{-1}(x_0)}\left[u\left(\alpha^k(\alpha^{-1}(x_0))\right)\right].$$

Considering (2), we can write:

$$T^{(k)}_{\alpha^{-1}(x_0)} = T^{-1}_{\alpha^{-1}(x_0)} \circ T^{-1}_{x_0} \circ \cdots \circ T_{\alpha^{k-2}(x_0)} = T^{-1}_{\alpha^{-1}(x_0)} \circ T^{(k-1)}_{x_0}, \text{ i.e.,}$$

$$\sigma_n^+(u, \alpha^{-1}(x_0)) = \sum_{k=0}^{n} T^{-1}_{\alpha^{-1}(x_0)}\left[T^{(k-1)}_{x_0}\left(u(\alpha_k{}^{-1}(x_0))\right)\right].$$

Counting the difference $\sigma_{n+p}^+(u, \alpha^{-1}(x_0)) - \sigma_n^+(u, \alpha^{-1}(x_0))$ and taking into account that $T^{-1}_{\alpha^{-1}}(x_0)$ is bounded ([2]), one sees, that $\{\sigma_n^+(u, \alpha^{-1}(x_0))\}_{n\in N}$ is a Cauchy-sequence, i.e., convergent. Passing to $\sigma_n^-(u, \alpha^{-1}(x_0)) = \sum_{k=-1}^{-n} T^{(k)}_{\alpha^{-1}(x_0)}\left[u(\alpha^{k-1}(x_0))\right]$ we observe that one can obtain with ease the relation: $T^{(k)}_{\alpha^{-1}(x_0)} = T^{-1}_{\alpha^{-1}(x_0)} \circ T^{(k-1)}_{x_0}$ which allows one to repeat the reasoning effected for

$$\{\sigma_n^+(u, \alpha^{-1}(x_0))\}_{n\in N}.$$

Thus $\alpha^{-1}(x_0) \in D$, i.e. the set D is saturated.

Theorem 2. *Let $\sigma(u, x)$ be an iterative series, convergent on a set $D \subseteq X$. Then the function $\sigma(u, x)$ is a solution of the equation* (1) *on the set D.*

Proof. Let us consider $x_0 \in D$; according to theorem 1: $\alpha(x_0) \in D$, thus the series is convergent in the point $\alpha(x_0)$ too. We have

$$\sigma(u, \alpha(x_0)) = \lim_{n\to\infty} \sum_{k=-n}^{n} T^{(k)}_{\alpha(x_0)}\left[u(\alpha^{k+1}(x_0))\right].$$

According to theorem 1: $T^{(k)}_{\alpha(x_0)} = T_{x_0} \circ T^{(k+1)}_{x_0}$ for any $k \in I$. From this relation we infer:

$$\sigma(u, \alpha(x_0)) = \lim_{n\to\infty} \sum_{k=-n}^{n} T_{x_0}\left[T^{(k+1)}_{x_0}\left(u(\alpha^{k+1}(x_0))\right)\right] = \lim_{n\to\infty} T_{x_0}\left[\sum_{k=-n}^{n} T^{(k+1)}_{x_0}\left(u(\alpha^{k+1}(x_0))\right)\right] =$$

$$= T_{x_0}\left[\lim_{n\to\infty} \sum_{k=-n}^{n} T^{(k+1)}_{x_0}\left(u(\alpha^{k+1}(x_0))\right)\right] = T_{x_0}\left[\lim_{n\to\infty} \sum_{i=-n+1}^{n+1} T^{(i)}_{x_0}\left(u(\alpha^{i}(x_0))\right)\right] =$$

$$= T_{x_0}(\sigma(u, x_0)) = T(x_0, \sigma(u, x_0)),$$

q.e.d.

Theorem 3. *Let the conditions* (i), (ii) *be fulfilled and in addition, let us suppose that for any point $x \in X$, the corresponding class is of the T_2-type* [1]. *Then for any solution f of the equation* (1), *there exists a function defined on X, $u: X \to Y$, such that:*

$$f(x) = \sigma(u, x). \tag{3}$$

Proof. Let S be a section of the set X (i. e., a set which contains from every class one, and only one point). Let us define:

$$u(x) = \begin{cases} f(x) & \text{if } x \in S, \\ \Theta & \text{if } x \notin S. \end{cases}$$

We shall point out, first of all, that $\sum_{k=-\infty}^{\infty} T_x^{(k)}[u(\alpha^k(x))]$ is convergent. For any $x \in X$ there exists a k_0, uniquely determined, such that $\alpha^{k_0}(x) \in S$; for $k \neq k_0$ we have $\alpha^k(x) \notin S$, from which one gets $u(\alpha^k(x)) = \Theta$ for $k \neq k_0$ and by virtue of the linearity of $T_x^{(k)}$:

$T_x^{(k)}[u(\alpha^k(x))] = \Theta$ for $k \neq k_0$. From this obviously results that $\sum_{k=-\infty}^{\infty} T_x^{(k)}[u(\alpha^k(x))]$ is convergent, thus it is a solution of the equation (1) on X (the point $x \in X$ was choosen to be arbitrary). Let us now point out that for any x there holds the relation (3). Indeed, if $x \in S$ then $\alpha^k(x) \notin S$ for $k \neq 0$, thus $\sigma(u, x) = T_x^0(u(x)) = E(u(x)) = u(x) = f(x)$. Having in view the theorem of uniquness from [1], one sees that relation (3) takes place on the entire set X.

Remark. According to the hypotheses from theorem 3, the general structure of the solutions of the equation (1) is given by the formula (3).

Application. Let us consider the particular case of the equation (1), having the form:

$$(1)' \qquad f[\alpha(x)] = \lambda(x) \cdot f(x)$$

where λ is a given function defined on the set X, taking values in the set of the invertible elements of the Banach-algebra Y, with unit element.

It is obvious that the function $T(x, y) = \lambda(x) \cdot y$ is a linear and bounded operator for any fixed x (we hawe $\|T(x, y)\| \leqq \|\lambda(x)\| \cdot \|y\|$). Furthermore T_x achieves a bijection of Y on to Y. Therefore for the equation (1)′ the condition (ii) is fulfilled. It is observed with ease that inthis instance an iterative series has the form: $\sum_{k=-\infty}^{\infty} p_k(x) \cdot u(\alpha^k(x))$ where

$$(2)' \qquad p_k(x) = \begin{cases} [\lambda(x)]^{-1} \cdot [\lambda(\alpha(x))]^{-1} \dots [\lambda(\alpha^{k-1}(x))]^{-1} & \text{for} \quad k > 0, \\ e \text{ (the unit element from } Y) & \text{for} \quad k = 0, \\ \lambda(\alpha^{-1}(x)) \cdot \lambda(\alpha^{-2}(x)) \dots (\lambda(\alpha^k(x))) & \text{for} \quad k < 0; \end{cases}$$

thus, assuming the hypotheses from theorem 3:

Any solution of the equation (1)′ has the form $\sum_{k=-\infty}^{\infty} p_k(x) \cdot u(\alpha^k(x))$, where the family $\{p_k(x)\}_{k \in I}$ is defined through the relation (2)′, and u is a function defined on X with values in Y. This is in fact a generalization of one of C. Popovici's [3] theorems.

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