A sub class of α-spiral functions

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1. Introduction

Let $G(\alpha)$ denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are regular and analytic in the unit disc $E=\{z:|z|<1\}$ with the property

$$\text{Re}\left[e^{i\alpha}\frac{zf'(z)}{f(z)}\right]>0, \quad z\in E \quad \text{and} \quad |\alpha|\leq\frac{\pi}{2}. \tag{1}$$

Functions satisfying conditions (1) are called α-spiral functions and it was shown by ŠPAČEK [3] that such functions are univalent in E. When $|\alpha|=\pi/2$, the function $f(z)$ reduces to the identity function z and hence we exclude such a case from our consideration.

In this paper we shall consider a sub-class $H(\alpha)$ of those functions in $G(\alpha)$ which satisfy the condition

$$\left|e^{i\alpha}\frac{zf'(z)}{f(z)}-(1+i\sin\alpha)\right|<1, \quad z\in E, \quad |\alpha|<\frac{\pi}{2}. \tag{2}$$

The class $H(0)$ coincides with the class of functions studied by R. SINGH in [2]. Hence on letting $\alpha=0$ in the theorems proved below, we shall get the results obtained earlier by R. Singh in [2].

2. A representation formula

Theorem 2.1. *Every function $f(z)\in H(\alpha)$ can be represented in the form*

$$f(z)=z\exp\left[e^{-i\alpha}\cos\alpha(2-\cos\alpha)\int_0^z\frac{\psi(\zeta)}{1-(1-\cos\alpha)\psi(\zeta)}\cdot\frac{d\zeta}{\zeta}\right] \tag{3}$$

where $\psi(\zeta)$ is a function satisfying Schwarz's Lemma [I, p. 165].

PROOF. Let

$$\varphi(z)=e^{i\alpha}\frac{zf'(z)}{f(z)}-1-i\sin\alpha, \tag{4}$$

then

$$\varphi(0) = \cos\alpha - 1. \tag{5}$$

Let

$$\psi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \varphi(0)\varphi(z)}, \tag{6}$$

then $\psi(0)=0$ and $|\psi(z)|<1$, therefore $\psi(z)$ satisfies the conditions of Schwarz's Lemma.

From (4), (5) and (6) we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1 + e^{-i\alpha}(e^{-i\alpha} + i\sin\alpha\cos\alpha)\psi(z)}{1 - (1-\cos\alpha)\psi(z)}. \tag{7}$$

From (7) we obtain after some simple computation and integration formula (3).

It is easy to see that every function $f(z)$ represented by formula (3) will belong to the class $H(\alpha)$.

3. Radius of starlikeness

Theorem 3.1. *I $f(z)\in H(\alpha)$ and $|z|=r$, $0<r<1$, then*

$$\frac{1-\cos\alpha(2-\cos\alpha)r + (1-\cos\alpha)(\cos 2\alpha + \sin^2\alpha\cos\alpha)r^2}{1-(1-\cos\alpha)^2r^2} \leqq \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \leqq \tag{8}$$
$$\leqq \frac{1+\cos\alpha(2-\cos\alpha)r + (1-\cos\alpha)(\cos 2\alpha + \sin^2\alpha\cos\alpha)r^2}{1-(1-\cos\alpha)^2r^2}.$$

These bounds are sharp.

Proof. The right hand side of (7) is subordinate to the function

$$\frac{1 + e^{-i\alpha}(e^{-i\alpha} + i\sin\alpha\cos\alpha)z}{1-(1-\cos\alpha)z}$$

which for any r, $0<r<1$, maps the disc $|z|\leqq r$ onto the disc with centre

$$\frac{1 + e^{-i\alpha}(e^{-i\alpha} + i\sin\alpha\cos\alpha)(1-\cos\alpha)r^2}{1-(1-\cos\alpha)^2r^2}$$

and radius

$$\frac{\cos\alpha(2-\cos\alpha)r}{1-(1-\cos\alpha)^2r^2}.$$

Therefore, we obtain

$$\left|\frac{zf'(z)}{f(z)} - \frac{1 + e^{-i\alpha}(e^{-i\alpha} + i\sin\alpha\cos\alpha)(1-\cos\alpha)r^2}{1-(1-\cos\alpha)^2r^2}\right| \leqq \frac{\cos\alpha(2-\cos\alpha)r}{1-(1-\cos\alpha)^2r^2}. \tag{9}$$

Inequalities in (8) now follow at once from (9).

The sharp bounds in (γ) on the left and on the right are attained on taking $\psi(z)=z$ in the function defined by (7), that is in the function.

$$(10)\qquad f(z)=z\big(1-(1-\cos\alpha)z\big)^{\frac{-\cos\alpha(2-\cos\alpha)e^{-i\alpha}}{1-\cos\alpha}}\quad\left(\text{where }\frac{f(z)}{z}=1\ \text{ at }\ z=0\right)$$

at the points $re^{i\Theta_1}$ and $re^{i\Theta_2}$ respectively where

$$(11)\qquad \tan\Theta_1=\frac{-\big(1-(1-\cos\alpha)^2r^2\big)\sin\alpha}{2(1-\cos\alpha)r-\big(1+(1-\cos\alpha)^2r^2\big)\cos\alpha},$$

and

$$(12)\qquad \tan\Theta_2=\frac{\big(1-(1-\cos\alpha)^2r^2\big)\sin\alpha}{2(1-\cos\alpha)r+\big(1+(1-\cos\alpha)^2r^2\big)\cos\alpha},\quad |z|=r.$$

Theorem 3.2. *If $f(z)\in H(\alpha)$ then $f(z)$ is starlike in $|z|<r_0$ where r_0 is the smaller of the two positive roots of the equation*

$$(13)\qquad 1-\cos\alpha(2-\cos\alpha)r+(1-\cos\alpha)(\cos 2\alpha+\sin^2\alpha\cos\alpha)r^2=0.$$

The result is sharp.

Proof. The theorem follows immediately from the left hand inequality in (8).

4. Bounds for $|f(z)|$

Theorem 4.1. *If $f(z)\in H(\alpha)$ and $|z|=r$, $0<r<1$, then*

$$(14)\qquad \begin{aligned} &r\left(\frac{1+Ar}{1-Ar}\right)^{-B/2A}\cdot\frac{1}{(1-A^2r^2)^{\cos\alpha/2A}}\leqq\\ &\leqq|f(z)|\leqq r\left(\frac{1+Ar}{1-Ar}\right)^{B/2A}\cdot\frac{1}{(1-A^2r^2)^{\cos\alpha/2A}}\end{aligned}$$

where $A=1-\cos\alpha,\quad B=\cos\alpha(2-\cos\alpha)>0\quad$ for $\quad|\alpha|<\pi/2$.

These bounds are sharp only when $\alpha=0$.

Proof. From (7) we have

$$(15)\qquad z\frac{d}{dz}\left(\log\frac{f(z)}{z}\right)=\frac{e^{-i\alpha}B\psi(z)}{1-A\varphi(z)},$$

or

$$(16)\qquad f(z)=z\exp\left[e^{-i\alpha}B\int_0^z\frac{\psi(\zeta)}{1-A\psi(\zeta)}\cdot\frac{d\zeta}{\zeta}\right].$$

Therefore,

$$(17)\qquad |f(z|=|z|\cdot\exp\left[B\int_0^r\operatorname{Re}\left(\frac{e^{-i\alpha}\psi(te^{i\theta})}{1-A\psi(te^{i\theta})}\right)\cdot\frac{dt}{t}\right].$$

Since $\frac{\psi(z)}{1-A\psi(z)}$ is subordinate to $\frac{z}{1-Az}$ which for any $r, 0<r<1$ maps the disc $|z|\leqq r$ onto the disc with centre $\frac{Ar^2}{1-A^2r^2}$ and radius $\frac{r}{1-A^2r^2}$ therefore we have

$$\left|\frac{\psi(z)}{1-A\psi(z)}-\frac{Ar^2}{1-A^2r^2}\right|\leqq\frac{r}{1-A^2r^2}, \tag{18}$$

which yields

$$\frac{A\cos\alpha r^2-r}{1-A^2r^2}\leqq \operatorname{Re}\left[\frac{e^{-i\alpha}\psi(z)}{1-A\psi(z)}\right]\leqq\frac{A\cos\alpha\cdot r^2+r}{1-A^2r^2}. \tag{19}$$

(17) together with right hand side of (19) yields

$$\begin{aligned}|f(z)| &\leqq r\exp\left[B\int_0^r\frac{A\cos\alpha\cdot t+1}{1-A^2t^2}\,dt\right]=\\ &=r\exp\left[\frac{B}{2A}\log\frac{1+Ar}{1-Ar}+\frac{\cos\alpha}{2A}\log\frac{1}{1-A^2r^2}\right]=\\ &=r\cdot\left(\frac{1+Ar}{1-Ar}\right)^{B/2A}\cdot\frac{1}{(1-A^2r^2)^{(\cos\alpha)/2A}}.\end{aligned} \tag{20}$$

Similarly we obtain the lower bound for $|f(z)|$.

Remark 1. Equalities on either side of (14) are attained at a point z, $|z|=r$ if and only if $\psi(z)=\varepsilon z$, $|\varepsilon|=1$. But, a computation shows that equalities will occur on either side of (19) at points $re^{i\Theta_1}$ and $re^{i\Theta_2}$, Θ_1, Θ_2 being functions of r. Consequently, the estimates made in (20) are not sharp in general because the integration is performed along a ray from the origin.

5. Coefficient estimates

If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in H(\alpha)$, then

$$|a_2|\leqq\cos\alpha(2-\cos\alpha),\quad |\alpha|<\pi/2, \tag{21}$$

$$|a_n|\leqq\frac{\cos\alpha(2-\cos\alpha)}{n-1}\quad\text{for}\quad n\geqq 2\quad\text{and}\quad \cos\alpha\leqq\frac{1}{3},\quad \alpha=0. \tag{22}$$

These bounds are sharp.

PROOF. Substituting the power series for $f(z)$ in (7) we get after some simplification

$$\sum_{k=2}^{\infty}(k-1)a_kz^k=\left[\sum_{k=1}^{\infty}\left(k(1-\cos\alpha)+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)\right)a_kz^k\right]\psi(z).$$

Rewritting it in the form

$$\sum_{k=2}^{n}(k-1)a_k z^k+\sum_{k=n+1}^{\infty}c_k z^k=\left[\sum_{k=1}^{n-1}\big(k(1-\cos\alpha)+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)\big)a_k z^k\right]\psi(z),$$

$\sum_{k=n+1}^{\infty}c_k z^k$ being absolutely and uniformly convergent on compact subsets of E. Putting $z=re^{i\Theta}$ and using the fact that $\psi(z)$ is bounded by 1, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=2}^{n}(k-1)a_k r^k e^{i\Theta k}+\sum_{k=n+1}^{\infty}c_k r^k e^{i\Theta k}\right|^2 d\Theta\leqq \tag{23}$$

$$\leqq\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{n-1}\big(k(1-\cos\alpha)+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)\big)a_k r^k e^{i\Theta k}\right|^2 d\Theta.$$

Applying Parseval's identity [1, p. 100] to (23) we get

$$\sum_{k=2}^{n}(k-1)^2|a_k|^2 r^{2k}+\sum_{k=n+1}^{\infty}|c_k|^2 r^{2k}\leqq \tag{24}$$

$$\leqq\sum_{k=1}^{n-1}|k(1-\cos\alpha)+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)|^2|ak|^2 r^{2k}.$$

Since the infinite series is non-negative and $0<r<1$, we can write

$$\sum_{k=2}^{n}(k-1)^2|a_k|^2\leqq\sum_{k=1}^{n-1}|k(1-\cos\alpha)+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)|^2|a_k|^2,$$

or

$$(n-1)^2|a_n|^2\leqq|1-\cos\alpha+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)|^2+$$

$$+\sum_{k=2}^{n-1}[|k(1-\cos\alpha)+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)|^2-(k-1)^2]|a_k|^2= \tag{25}$$

$$=\cos^2\alpha(2-\cos\alpha)^2-\cos\alpha(2-\cos\alpha)\sum_{k=2}^{n-1}[k^2-2k(\sin^2\alpha+\cos\alpha)+\sin^2\alpha]|a_k|^2.$$

An easy calculation shows that $k^2-2k(\sin^2\alpha+\cos\alpha)+\sin^2\alpha$ is an increasing function of k and remains positive for $k\geqq 3$, $|\alpha|<\pi/2$. For $k=2$, $k^2-2k(\sin^2\alpha+\cos\alpha)+$ $+\sin^2\alpha=(1-\cos\alpha)(1-3\cos\alpha)$ which is non-negative for $\alpha=0$ or for $\cos\alpha\leqq 1/3$. Hence for $\alpha=0$ or for $\cos\alpha\leqq 1/3$ we obtain from (25), $(n-1)^2|a_n|^2\leqq\cos^2\alpha(2-\cos\alpha)^2$ which yields (22).

Also from (25) we obtain for $|\alpha|<\pi/2$

$$|a_2|\leqq\cos\alpha(2-\cos\alpha)$$

which is (21).

For $1 > \cos\alpha > 1/3$ and $n \geqq 3$ on using (21) we obtain from (25)

$$(n-1)^2|a_n|^2 \leqq \cos^2\alpha(2-\cos\alpha)^2 + \cos\alpha(2-\cos\alpha)(1-\cos\alpha)(3\cos\alpha-1)|a_2|^2 \leqq$$
$$\leqq \cos^2\alpha(2-\cos\alpha)^2[1+\cos\alpha(2-\cos\alpha)(1-\cos\alpha)(3\cos\alpha-1)]$$

or

$$(26)\quad |a_n| \leqq \frac{\cos\alpha(2-\cos\alpha)\sqrt{1+\cos\alpha(2-\cos\alpha)(1-\cos\alpha)(3\cos\alpha-1}}{n-1} \quad \text{for} \quad n \geqq 3.$$

For any fixed $n \geqq 2$ and for $\alpha = 0$ or $\cos\alpha \leqq 1/3$, the function $f(z)$ defined by

$$\frac{zf'(z)}{f(z)} = \frac{1+e^{-i\alpha}(e^{-i\alpha}+i\sin\alpha\cos\alpha)z^{n-1}}{1-(1-\cos\alpha)z^{n-1}}$$

gives sharp estimates.

Remark 2. When $1/3 < \cos\alpha < 1$, $n \geqq 3$, we have not been able so far to obtain sharp bounds for $|a_n|$.

References

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