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A Frobenius-type theorem for supersolvable groups

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Abstract. Frobenius' Theorem for p-nilpotent groups is one of the most fundamental theorems in finite group theory. In this paper a Frobenius-type Theorem for supersolvable groups is given by applying strictly p-closed groups, and some applications are obtained.

Throughout, all groups mentioned are assumed to be finite groups. The terminology and notations employed agree with standard usage.

Let p be a prime. A group G is said to be strictly p-closed whenever G_p, a Sylow p-subgroup of G, is normal in G with G/G_p Abelian of exponent dividing $p-1$.

Let P be a Sylow p-subgroup of a group G; Frobenius' Theorem [1, Theorem 10.3.2] states that: a group G is p-nilpotent, if and only if $N_G(P_1)/C_G(P_1)$ is a p-group for every subgroup P_1 of P. If the condition that $N_G(P_1)/C_G(P_1)$ is a p-group is replaced by the weaker condition that $N_G(P_1)/C_G(P_1)$ is a strictly p-closed group, we can obtain a generalization of Frobenius' Theorem for supersolvable groups.

First we prove the following

Theorem 1. *Let G be a p-solvable group, N a normal subgroup of G such that G/N is a p-supersolvable group. If $N_G(P)/C_G(P)$ is strictly p-closed for every p-subgroup P of N, then G is p-supersolvable.*

Proof. Let K be a minimal normal subgroup of G contained in N. Then K is an elementary Abelian p-group or a p'-group since G is a p-solvable group. Set $\overline{G} = G/K$, and $\overline{N} = N/K$. If K is an elementary Abelian p-group, then, for every p-subgroup $\overline{P} = P/K$ of $\overline{N}$, P is a p-subgroup of N, and so $N_G(P)/C_G(P)$ is strictly p-closed. Since the quotient group of a strictly p-closed group is also a strictly p-closed group, $(N_G(P)/K)/(C_G(P)K/K)$ is a strictly p-closed group. It follows from $N_G(P)/K = N_{G/K}(P/K)$ and $C_{G/K}(P/K) \geq C_G(P)K/K$ that $N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$ is a strictly p-closed group. If K is a p'-group, then, for every p-subgroup $\overline{P} = H/K$ of $\overline{N}$, $H = PK$, where $P \in \mathrm{Syl}_p H$. By the condition $N_G(P)/C_G(P)$ is strictly p-closed,and so $N_G(P)K/C_G(P)K$ is also strictly p-closed. It is clear that $C_{\overline{G}}(\overline{P}) \geq C_G(P)K/K$. Using [3, Theorem 3.16] $N_{\overline{G}}(\overline{P}) = N_G(P)K/K$ we have that $N_{\overline{G}}(\overline{P})/C_{\overline{G}}(\overline{P})$ is strictly p-closed. Hence we conclude by induction that G/K is p-supersolvable.

If K is a p'-group, then G is p-supersolvable. If K is an elementary Abelian p-group, set $C = C_G(K)$. By the condition G/C is strictly p-closed. Let $A/C \in \mathrm{Syl}_p(G/C)$, then $A/C \triangleleft G/C$, and the semidirect product $A/C \ltimes K$ is a p-group. Hence $Z(A/C \ltimes K) \cap K \neq 1$. Since G/C can act on $Z(A/C \ltimes K) \cap K$, by conjugation and since the action of G/C on K is irreducible we have $Z(A/C \ltimes K) \cap K = K$. Hence the action of A/C on K is trivial and $A/C = 1$. Therefore G/C is Abelian of exponent dividing $p-1$. By [2, Theorem I.1.4] $|K| = p$, and G is p-supersolvable. The proof of Theorem 1 is complete.

Theorem 2. *Let N be a normal subgroup of a group G, and G/N be a supersolvable group. Then G is a supersolvable group if and only if for every prime $p \mid |N|, N_G(P)/C_G(P)$ is a strictly p-closed group for every p-subgroup P of N.*

The proof of Theorem 2 needs the following

Lemma 1. *Let P be a p-subgroup of a group G, and $N_G(P)/C_G(P)$ a strictly p-closed group. If H is a subgroup of G, and $P \leq H$, then $N_H(P)/C_H(P)$ is a strictly p-closed group too.*

Proof. Since $N_H(P) = H \cap N_G(P)$ and $C_H(P) = H \cap C_G(P)$, we have

$$N_H(P)/C_H(P) = H \cap N_G(P)/H \cap C_G(P) \simeq [H \cap N_G(P)]C_G(P)/C_G(P).$$

Noticing that subgroups of a strictly p-closed group are strictly p-closed groups, $N_H(P)/C_H(P)$ is strictly p-closed.

Proof of Theorem 2. Assume first that G is a supersolvable group. Let p be a prime, P a p-subgroup of $N, H = N_G(P)$. Since $P \triangleleft H$, we have a chief series of H passing through P:

$$1 = P_0 < P_1 < \cdots < P_S = P \leq \cdots \leq H.$$

As a subgroup of the supersolvable group G, H itself is supersolvable, and so $|P_j/P_{j-1}| = p$ $(j = 1, 2, \ldots, s)$. By [2, Theorem I.1.4]

$$\mathrm{Aut}_H(P_j/P_{j-1}) \simeq H/C_H(P_j/P_{j-1})$$

is Abelian of exponent dividing $p-1$. Set $L = \bigcap_{j=1}^{s} C_H(P_j/P_{j-1})$ and $C = C_G(P)$, then $L \lhd H$ and H/L is also Abelian of exponent dividing $p-1$, and moreover, $L \geq C$. We claim that L/C is a p-group. Suppose to the contrary that some $Cx \in L/C$ has order n relatively prime to p. Let $\alpha \in \mathrm{Aut}(P)$ be the automorphism induced by x, i.e., $\alpha(g) = x^{-1}gx$ $(g \in P)$, then the order of α in $\mathrm{Aut}(P)$ divides n, hence it is also relatively prime to p. Also note that $x \in L$ implies $[P_j, \alpha] \leq P_{j-1}$ for $1 \leq j \leq s$, so that [2, Lemma I.1.11] applies to show α is trivial. Hence so is Cx too, proving the claim. It follows that $N_G(P)/C_G(P)$ is strictly p-closed with Sylow p-subgroup L/C.

Suppose now that for every prime $p \bigm| |N|, N_G(P)/C_G(P)$ is a strictly p-closed group for every p-subgroup P of N. Let K be a minimal normal subgroup of G contained in N. Then K is a p-group for some prime p. In fact, assume that p is the smallest prime dividing $|K|$; by Lemma 1 and $(p-1, |K|) = 1, N_K(P)/C_K(P)$ is a p-group for every p-subgroup P of K. Using Frobenius' Theorem [1, Theorem 10.3.2], K has a normal p-complement, say L. Noticing that $L \lhd G$, $L < K$ and that K is a minimal normal subgroup of G, we have $L = 1$, and hence K is an elementary Abelian p-group.

Set $\overline{G} = G/K$ and $\overline{N} = N/K$. Similarly to the proof of Theorem 1 we have that for every prime $q \bigm| |\overline{N}|$, $N_{\overline{G}}(\overline{R})/C_{\overline{G}}(\overline{R})$ is strictly q-closed for every q-subgroup $\overline{Q}$ of $\overline{N}$. Hence we conclude by induction that G/K is supersolvable. By the condition and Theorem 1 G is p-supersolvable. Noticing that K is a minimal normal p-subgroup of G, we have that K is a cyclic group of order p. It follows that G is supersolvable. The proof of Theorem 2 is complete.

Corollary 1. *A group G is supersolvable if and only if, for every prime $p \bigm| |G|$, $N_G(P)/C_G(P)$ is strictly p-closed for every p-subgroup P of G.*

Theorem 3. *Let N be a normal subgroup of a group G, and G/N a supersolvable group. Then G is supersolvable if and only if, for every prime $p \bigm| |N|$, $[N_G(P)/C_G(P)]'$ and $[N_G(P)/C_G(P)]^{p-1}$ are p-groups for every p-subgroup P of N.*

From Theorem 2 and the following Lemma 2 Theorem 3 is immediate.

Lemma 2. *A group G is strictly p-closed if and only if G' and G^{p-1} are p-groups.*

Proof. If G is strictly p-closed, then G/G_p is Abelian, where $G_p \in \mathrm{Syl}_p\, G$. Hence $G' \le G_p$ and G' is a p-group. It follows from the exponent of G/G_p dividing $p-1$ that $g^{p-1} \in G_p$ for every $g \in G$, therefore G^{p-1} is also a p-group.

Suppose now that G' and G^{p-1} are p-groups. Let $G_p \in \mathrm{Syl}_p\, G$. Since $G' \lhd G$, we have $G' \le G_p$ and so $G_p \lhd G$ and G/G_p is Abelian. By using that G^{p-1} is a p-group we have $G^{p-1} \le G_p$. Hence G/G_p is Abelian of exponent dividing $p-1$.

Corollary 2. *A group G is supersolvable if and only if, for every prime $p \mid |G|, [N_G(P)/C_G(P)]'$ and $[N_G(P)/C_G(P)]^{p-1}$ are p-groups for every p-subgroup P of G.*

As an application of Theorem 2, we prove the following

Theorem 4. *Let N be a normal subgroup of a group G, and G/N be a supersolvable group. If every minimal subgroup of N is pronormal in G, and either the Sylow 2-subgroups of N are Abelian or every cyclic subgroup of N of order 4 is pronormal in G, then G is supersolvable.*

The proof of Theorem 4 needs the following

Lemma 3. *Let $A_1, A_2, \ldots, A_S;\ B_1, B_2, \ldots, B_S$ be subgroups of the group G, and $B_i \lhd A_i, (i = 1, 2, \ldots, s)$. If A_i/B_i is Abelian of exponent dividing m, then $(A_1 \cap A_2 \cap \cdots \cap A_s)/(B_1 \cap B_2 \cap \cdots \cap B_s)$ is also Abelian of exponent dividing m.*

Proof. We only prove Lemma 3 when $s = 2$. Clearly $B_1 \cap B_2 \lhd A_1 \cap A_2$. For any $g_1, g_2 \in A_1 \cap A_2$, since A_1/B_1 and A_2/B_2 are Abelian and $g_1(B_1 \cap B_2) = g_1B_1 \cap g_1B_2$, we have $g_1g_2(B_1 \cap B_2) = g_2g_1(B_1 \cap B_2)$, i.e., $A_1 \cap A_2/B_1 \cap B_2$ is Abelian. From $g_1^m \in B_1, g_1^m \in B_2$ we have $g_1^m \in B_1 \cap B_2$. Hence the exponent of $A_1 \cap A_2/B_1 \cap B_2$ divides m.

Proof of Theorem 4. For any prime $p \mid |N|$, if P is a subgroup of N of order p, then $N_G(P)/C_G(P)$ is Abelian of exponent dividing $p-1$ since $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$. Hence $N_G(P)/C_G(P)$ is strictly p-closed. If P is a cyclic subgroup of N of order 4, it follows from $|\mathrm{Aut}(P)|=2$ that $N_G(P)/C_G(P)$ is Abelian of exponent dividing 2. Hence $N_G(P)/C_G(P)$ is strictly 2-closed.

Let A be any p-subgroup of N, and x be an element of A of order p. Then $\langle x \rangle$ is subnormal in $N_G(A)$. Using [1, exercise 10.3.3] $\langle x \rangle \lhd N_G(A)$.

Since $\Omega_1(A) \lhd N_G(A) = H$, $C_H(\Omega_1(A)) \lhd H$, it is clear that $C = C_G(A) \leq C_H(\Omega_1(A))$. We claim that $C_H(\Omega_1(A))/C$ is a p-subgroup of H/C if $p \neq 2$, or $p = 2$ and A is Abelian. In fact, let $gC \in C_H(\Omega_1(A))/C$ and the order of gC be a p'-number. Noticing that $\langle gC \rangle$ can act on A by conjugation, and that the action of $\langle gC \rangle$ on $\Omega_1(A)$ is trivial, the action of $\langle gC \rangle$ on A is trivial by [3, Theorem 7.26] if $p \neq 2$ or by [4, Theorem 5.2.4] if $p = 2$ and A is Abelian. Hence $gC = C$, i.e., $C_H(\Omega_1(A))/C$ is a p-group. Noticing that $C_H(\Omega_1(A)) = \bigcap_{x \in \Omega_1(A)} (C_H(\langle x \rangle))$, $H \subseteq \bigcap_{x \in \Omega_1(A)} N_H(\langle x \rangle)$ and that $N_H(\langle x \rangle)/C_H(\langle x \rangle)$ is Abelian of exponent dividing $p - 1$ (when x has order p), $H/C_H(\Omega_1(A))$ is Abelian of exponent dividing $p-1$ by Lemma 3. Hence $H/C = N_G(A)/C_G(A)$ is strictly p-closed if $p \neq 2$ or $p = 2$ and A is Abelian.

If A is a 2-subgroup of N and A is not Abelian, by considering the subgroup $\Omega_2(A)$ and using [3, Theorem 7.26], similar to the above proof we have that $C_H(\Omega_2(A))/C_G(A)$ is a 2-group, and that $H/C_H(\Omega_2(A))$ is Abelian of exponent dividing 2. Hence $H/C = N_G(A)/C_G(A)$ is a 2-group, and so strictly 2-closed. By Theorem 1 G is supersolvable. The proof of Theorem 4 is complete.

Remark. The statement of Theorem 4 for the case when N has odd order has been proved by M. Asaad in [5].

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