A note on nonisomorphic reverse Steiner quasigroups

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1. Introduction

A *Steiner quasigroup* is a quasigroup satisfying the identities $x(xy)=y$, $(yx)x=y$, and $x^2=x$. It is well known that the spectrum for Steiner quasigroups is the set of all positive integers $n\equiv 1$ or 3 (mod 6). A Steiner quasigroup is said to be *reverse* provided that its automorphism group contains an involution fixing exactly one element. Such an automorphism is called a *reverse automorphism*. Recently, the combined work of A. ROSA [4], J. DOYEN [1], and L. TEIRLINCK [6] established the fact that the spectrum for reverse Steiner quasigroups consists of all positive integers $n\equiv 1, 3, 9$, or 19 (mod 24). In [2], C. C. LINDNER gave a new construction for reverse Steiner quasigroups in order to construct large numbers of nonisomorphic reverse Steiner quasigroups of a given order. The purpose of this note is to give a still different method of constructing reverse Steiner quasigroups and to use this construction to construct nonisomorphic reverse Steiner quasigroups. One advantage of the construction given in this note is that it can be used to construct nonisomorphic reverse Steiner quasigroups of smaller orders than the construction given by Lindner in [2].

2. Construction of reverse Steiner quasigroups

Let $(Q, 0)$ and $(V, \odot)$ be reverse Steiner quasigroups with reverse automorphisms α and β respectively. We can partition the elements of V which are moved by β into sets A and B such that $v\in A$ if and only if $v\beta\in B$. For each $v\in A$ let $O(v)$ be a binary operation on Q such that $(Q, O(v))$ is a Steiner quasigroup. For each $v\in B$ define a binary operation $O(v)$ on Q by $pO(v)q=((p\alpha)O(v\beta)(q\alpha))\alpha$. It is a routine matter to show that Q equipped with this operation is a Steiner quasigroup. We denote by $(Q\times V, \overline{\oplus})$ the ordinary direct product of the reverse Steiner quasigroups $(Q, 0)$ and $(V, \odot)$. The mapping γ defined by $(p, v)\gamma=(p\alpha, v\beta)$ is easily seen to be a reverse automorphism so that $(Q\times V, \overline{\oplus})$ is in fact a reverse Steiner quasigroup. It is well known that if any subquasigroup of a Steiner quasigroup is replaced by any Steiner quasigroup on these same symbols that the resulting quasigroup is still a Steiner quasigroup. Hence if we replace the $|V|-1$ disjoint subquasigroups $(Q\times\{v\}, \overline{\oplus})$ with the quasigroups $(Q\times\{v\}, O(v))$, all $v\in A\cup B$, the result is still a Steiner quasigroup which we will denote by $(Q\times V, \oplus)$. We now show that $(Q\times V, \oplus)$ is a reverse Steiner quasigroup.

Theorem 1. *The Steiner quasigroup $(Q\times V, \oplus)$ constructed above is a reverse Steiner quasigroup.*

PROOF. As previously mentioned, the mapping γ defined by $(p, v)\gamma=(p\alpha, v\beta)$ is a reverse automorphism of the direct product $(Q\times V, \overline{\oplus})$. Since $(p, v)\overline{\oplus}(q, w)=(p, v)\oplus(q, w)$, whenever $v\neq w$ or $v=w\notin A\cup B$, it follows that $((p, v)\oplus(q, w))\gamma=(p, v)\gamma\oplus(q, w)\gamma$ in either of these cases. Thus, we need only check the cases where $v=w\in A\cup B$. There are two cases to consider.

1) $v\in A$. In this case

$$((p, v)\oplus(q, v))\gamma = (pO(v)q, v)\gamma = ((pO(v)q)\alpha, v\beta).$$

But

$$(p, v)\gamma\oplus(q, v)\gamma = (p\alpha, v\beta)\oplus(q\alpha, v\beta) = (((p\alpha)\alpha O((v\beta)\beta)(q\alpha)\alpha)\alpha, v\beta),$$

since $v\beta\in B$, $= ((pO(v)q)\alpha, v\beta)$. Hence we have equality.

2) $v\in B$. In this case

$$((p, v)\oplus(q, v))\gamma = (((p\alpha)O(v\beta)(q\alpha))\alpha, v)\gamma =$$

$$= ((((p\alpha)O(v\beta)(q\alpha))\alpha)\alpha, v\beta) = ((p\alpha)O(v\beta)(q\alpha), v\beta).$$

On the other hand,

$$(p, v)\gamma\oplus(q, v)\gamma = (p\alpha, v\beta)\oplus(q\alpha, v\beta) = ((p\alpha)O(v\beta)(q\alpha), v\beta),$$

since $v\beta\in A$. Again we have equality.

Combining cases (1) and (2) along with the preceeding remarks completes the proof of the theorem.

3. Construction of nonisomorphic reverse Steiner quasigroups

The following theorem is similar to theorems found in [2], [3], and [5].

Theorem 2. *Let $(Q, 0)$ and $(V, \odot)$ be reverse Steiner quasigroups such that $|Q|>|V|$ and $(V, \odot)$ contains no nontrivial subquasigroup whose order divides $|Q|$. If $(Q\times V, \oplus)$ is any one of the reverse Steiner quasigroups constructed in Theorem 1, then the only subquasigroups of $(Q\times V, \oplus)$ of order $|Q|$ are the subquasigroups $(Q\times\{v\}, \oplus)$ for each v in V.*

PROOF. Let $(Q, 0)$, $(V, \odot)$, and $(Q\times V, \oplus)$ be as in the statement of the theorem and let K be any subquasigroup of $(Q\times V, \oplus)$ of order $|V|$. Let $V'=\{v\in V|(q, v)\in K\}$ for some q in Q. Then $(V', \odot)$ is a subquasigroup of $(V, \odot)$. For each v in V' set $Q_v=\{q\in Q|(q, v)\in K\}$. It is not difficult to see that $|Q_v|=|Q_w|$ for all $v\neq w\in V'$. Since the sets Q_v and Q_w are disjoint, $v\neq w$, we must have $|Q|=t\cdot|V'|$ where $t=|Q_v|=|Q_w|$. Since $|Q|$ is not divisible by the order of any nontrivial subquasigroup of $(V, \odot)$ it follows that $|V'|=1$ and so $K=Q\times\{v\}$ for some v in V. Hence the only subquasigroups of $(Q\times V, \oplus)$ of order $|Q|$ are the subquasigroups $(Q\times\{v\}, \oplus)$, all v in V. This completes the proof of the theorem.

Let n and t be positive integers. We will denote by P_n^t the number of t-tuples of integers $(x_1, x_2, ..., x_t)$, where

$$x_1+x_2+\cdots+x_t = n \quad \text{and} \quad 0 \leqq x_i \leqq n, \quad i = 1, 2, \ldots, t.$$

Theorem 3. *Let $(Q, 0)$ and $(V, \odot)$ be reverse Steiner quasigroups such that $|Q|>|V|$ and $(V, \odot)$ contains no nontrivial subquasigroup whose order divides $|Q|$. If there are t nonisomorphic Steiner quasigroups of order $|Q|$, the there are at least $P^t_{\frac{v-1}{2}}$ nonisomorphic reverse Steiner quasigroups of order vq.*

Proof. The proof of this theorem is an immediate consequence of Theorem 2 along with the observation that for each $v \in A$ the quasigroups $(Q, O(v))$ and $(Q, O(v\beta))$ are isomorphic.

Remark. Although the Steiner quasigroups $(Q, 0)$ and $(V, \odot)$ must be reverse in the statement of Theorem 3, the t nonisomorphic Steiner quasigroups of order $|Q|$ are *not necessarily* reverse.

Now denote by $N(t)$ the number of nonisomorphic Steiner quasigroups of order t.

Corollary 4. If $t \equiv 1$ or 19 (mod 24) there are at least $N(t)$ nonisomorphic reverse Steiner quasigroups of order $3t$.

Proof. The proof follows from Theorem 3 by noting that 3 does not divide $t \equiv 1$ or 19 (mod 24) and that $P_1^t = N(t)$.

Examples. The smallest order that the construction for reverse Steiner quasigroups given in [2] can be used for to construct nonisomorphic reverse Steiner quasigroups is 171. The results in this paper can be used to construct nonisomorphic reverse Steiner quasigroups of smaller orders as the following two examples show. R. M. Wilson (unpublished) has shown that $N(19) \geqq 8894$ and $N(25) \geqq 10^{14}$. Hence Corollary 4 gives at least 8894 nonisomorphic reverse Steiner quasigroups of order 57 and at least 10^{14} nonisomorphic reverse Steiner quasigroups of order 75.

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