Axiomatic foundation of the n-dimensional Möbius geometry

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1. Introduction

G. EWALD [5] gave a simple system of axioms for the Möbius plane. In that plane there are only two fundamental concepts: the fundamental elements are circles and the fundamental relation is the relation of orthogonality of two circles. Another system of axioms for the Möbius plane was given in [8]. The analogous axiomatic construction of the n-dimensional Möbius space will be given here. Our axiomatic theory has only one set of fundamental elements, one fundamental relation and four axioms. Another system of axioms for the n-dimensional Möbius space was given by H. MÄURER [6].

2. Axioms

Let S be any nonempty set of elements which we call hyperspheres and let "$\perp$" be a binary relation on it. If for hyperspheres a, b the relation $a \perp b$ is valid, then we say that the hypersphere a is orthogonal to the hypersphere b. Let n be any natural number.

Definition 1. An ordered $(n+2)$-tuple $\mathfrak{A}=(a_1, a_2, \ldots, a_{n+2})$ of hyperspheres is called an $(n+1)$-simplexoid iff there is an ordered $(n+2)$-tuple $\mathfrak{B}=(b_1, b_2, \ldots, b_{n+2})$ of hyperspheres such that

$$(1) \qquad (\forall \alpha, \beta \in \{1, 2, \ldots, n+2\})[a_\alpha \perp b_\beta \Leftrightarrow \alpha \neq \beta].$$

The notion of simplexoid is due to M. ESSER [4].
From Definition 1 it follows immediately

Theorem 1. *If* $(a_1, a_2, \ldots, a_{n+2})$ *is an* $(n+1)$*-simplexoid and* $\alpha_1, \alpha_2, \ldots, \alpha_{n+2}$ *is any permutation of the indices* $1, 2, \ldots, n+2$, *then* $(a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_{n+2}})$ *is an* $(n+1)$*-simplexoid too.*

Theorem 2. *If* $(a_1, a_2, \ldots, a_{n+2})$ *is an* $(n+1)$*-simplexoid, then* $a_1, a_2, \ldots, a_{n+2}$ *are different hyperspheres.*

PROOF. Let $\alpha, \beta \in \{1, 2, \ldots, n+2\}, \alpha \neq \beta$. By Definition 1 there is an ordered $(n+2)$-tuple $(b_1, b_2, \ldots, b_{n+2})$ of hyperspheres such that the relation (1) holds. Then $a_\beta \perp b_\alpha$. If $a_\alpha = a_\beta$ were valid, then $a_\alpha \perp b_\alpha$ would follow, which is impossible. Hence $a_\alpha \neq a_\beta$.

Definition 2. A set $B \subseteq S$ of hyperspheres is called an independent set with respect to the relation of orthogonality or shortly $\perp$-independent set iff there is an $(n+1)$-simplexoid $(a_1, a_2, \ldots, a_{n+2})$ such that $B \subseteq \{a_1, a_2, \ldots, a_{n+2}\}$.

From this definition follows

Theorem 3. *Every subset of an $\perp$-independent set of hyperspheres is an $\perp$-independent set.*

Definition 3. We say that the set $A \subseteq S$ of hyperspheres is orthogonal to the set $B \subseteq S$ of hyperspheres and write $A \perp B$ iff $a \perp b$ for $\forall a \in A$ and $\forall b \in B$. In particular if $B = \{b\}$ is a singleton we say that the set A of hyperspheres is orthogonal to the hypersphere b and write $A \perp b$, and if $A = \{a\}$ is a singleton we say that the hypersphere a is orthogonal to the set B of hyperspheres and write $a \perp B$.

Definition 4. The structure $(S, \perp)$ is called an n-dimensional Möbius space iff:

S1. $(\forall a, b \in S)[a \perp b \Rightarrow b \perp a]$.

S2. $(\forall a_1, a_2, \ldots, a_{n+1} \in S)(\exists b \in S)\{a_1, a_2, \ldots, a_{n+1}\} \perp b$.

S3. $(\exists a_1, a_2, \ldots, a_{n+2} \in S)[b \in S \Rightarrow \neg(\{a_1, a_2, \ldots, a_{n+2}\} \perp b)]$.

S4. If $A = \{a_1, a_2, \ldots, a_{n+1}\}$ is an $\perp$-independent then

$$(\forall b, c \in S)[A \perp \{b, c\} \Rightarrow b = c].$$

Theorem 4. *If $(a_1, a_2, \ldots, a_{n+2})$ is an $(n+1)$-simplexoid, then there is one and only one ordered $(n+2)$-tuple $\mathfrak{B} = (b_1, b_2, \ldots, b_{n+2})$ of hyperspheres such that the relation (1) is valid. Then $\mathfrak{B}$ is also an $(n+1)$-simplexoid.*

PROOF. Let $(a_1, a_2, \ldots, a_{n+2})$ is an $(n+1)$-simplexoid. By Definition 1 there is an ordered $(n+2)$-tuple $\mathfrak{B} = (b_1, b_2, \ldots, b_{n+2})$ of hyperspheres such that (1) holds. But, by S1, from (1) it follows

$$(\forall \alpha, \beta \in \{1, 2, \ldots, n+2\})[b_\beta \perp a_\alpha \Leftrightarrow \alpha \neq \beta],$$

and by Definition 1 $\mathfrak{B}$ is an $(n+1)$-simplexoid. Suppose that there is an $(n+1)$-simplexoid $\mathfrak{B}' = (b_1', b_2', \ldots, b_{n+2}')$ such that

$$(2) \qquad (\forall \alpha, \beta \in \{1, 2, \ldots, n+2\})[a_\alpha \perp b_\beta' \Leftrightarrow \alpha \neq \beta].$$

From (1) and (2) it follows

$$(3) \qquad (\forall \alpha \in \{1, 2, \ldots, n+2\})\{a_1, \ldots, a_{\alpha-1}, a_{\alpha+1}, \ldots, a_{n+2}\} \perp \{b_\alpha, b_\alpha'\}.$$

On the other hand, by Definition 2 we get that

$$(4) \qquad (\forall \alpha \in \{1, 2, \ldots, n+2\})\{a_1, \ldots, a_{\alpha-1}, a_{\alpha+1}, \ldots, a_{n+2}\} \text{ is an } \perp\text{-independent set.}$$

By S4 from (3) and (4) it follows

$$(\forall \alpha \in \{1, 2, \ldots, n+2\}) b_\alpha = b'_\alpha,$$

i.e. $\mathfrak{B} = \mathfrak{B}'$, and the theorem is proved.

Definition 5. Two $(n+1)$-simplexoides $\mathfrak{A} = (a_1, a_2, \ldots, a_{n+2})$, $\mathfrak{B} = (b_1, b_2, \ldots, b_{n+2})$ will be called dual $(n+1)$-simplexoides and will be written $\mathfrak{A} = d(\mathfrak{B})$ or $\mathfrak{B} = d(\mathfrak{A})$ iff the relation (1) holds.

From Definition 1 and S1 follows

Theorem 5. $(\forall A, B \subseteq S)[A \perp B \Rightarrow B \perp A]$; $(\forall a \in S)(\forall B \subseteq S)[a \perp B \Leftrightarrow B \perp a]$.

3. $\perp$-independent sets

Theorem 6. *The set $A = \{a_1, a_2, \ldots, a_{n+2}\}$ of hyperspheres is an $\perp$-independent set iff there is no hypersphere b such that $A \perp b$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_{n+2}\}$ be the given set of hyperspheres. Suppose that

$$b \in S \Rightarrow \neg (A \perp b). \tag{5}$$

By S2 there are hyperspheres $b_1, b_2, \ldots, b_{n+2}$ such that

$$(\forall \alpha \in \{1, 2, \ldots, n+2\}) \{a_1, \ldots, a_{\alpha-1}, a_{\alpha+1}, \ldots, a_{n+2}\} \perp b_\alpha.$$

If for some $\alpha \in \{1, 2, \ldots, n+2\}$ it were $a_\alpha \perp b_\alpha$, then it should be $A \perp b_\alpha$, contrary to hypothesis (5). Therefore (1) holds, i.e. $\mathfrak{A} = (a_1, a_2, \ldots, a_{n+2})$ is an $(n+1)$-simplexoid and hence A is an $\perp$-independent set. Conversely, let A be an $\perp$-independent set, i.e. let $\mathfrak{A} = (a_1, a_2, \ldots, a_{n+2})$ be an $(n+1)$-simplexoid. Furthermore let $\mathfrak{B} = d(\mathfrak{A})$, $\mathfrak{B} = (b_1, b_2, \ldots, b_{n+2})$. By Definition 5 and S4 b_1 is the only hypersphere such that $\{a_2, \ldots, a_{n+2}\} \perp b_1$. By Definition 5 $\neg (a_1 \perp b_1)$, and the relation (5) follows.

From S3 and Theorem 6 we get

Theorem 7. *There are hyperspheres $a_1, a_2, \ldots, a_{n+2}$ such that $\{a_1, a_2, \ldots, a_{n+2}\}$ is an $\perp$-independent set.*

Theorem 8. *If $(a_1, a_2, \ldots, a_{n+2})$, $(b_1, b_2, \ldots, b_{n+2})$ are dual $(n+1)$-simplexoides and c_1 is a hypersphere such that $\neg (c_1 \perp b_1)$, then $(c_1, a_2, \ldots, a_{n+2})$ is an $(n+1)$-simplexoid.*

Proof. By Definition 5 and S· b_1 is the unique hypersphere such that $\{a_2, \ldots, a_{n+2}\} \perp b_1$, and as is by the hypothesis $\neg (c_1 \perp b_1)$, it follows that

$$b \in S \Rightarrow \neg (\{c_1, a_2, \ldots, a_{n+2}\} \perp b).$$

By Theorem 6 $\{c_1, a_2, \ldots, a_{n+2}\}$ is an $\perp$-independent set, i.e. $(c_1, a_2, \ldots, a_{n+2})$ is an $(n+1)$-simplexoid.

Theorem 9. *For $\forall c_1 \in S$, $\{c_1\}$ is an $\perp$-independent set.*

PROOF. By Theorem 7 there is an $(n+1)$-simplexoid $\mathfrak{A}=(a_1, a_2, \ldots, a_{n+2})$. Let $\mathfrak{B}=d(\mathfrak{A})$, $\mathfrak{B}=(b_1, b_2, \ldots, b_{n+2})$. Then $B=\{b_1, b_2, \ldots, b_{n+2}\}$ is an $\perp$-independent set, and from Theorem 6 follows

$$c\in S \Rightarrow \neg\, (c\perp B).$$

Therefore there is a hypersphere in B which is not orthogonal to c_1. Because of the symmetry we may suppose $\neg(c_1\perp b_1)$, and then, by Theorem 8 $(c_1, a_2, \ldots, a_{n+2})$ is an $(n+1)$-simplexoid, i.e. $\{c_1\}$ is an $\perp$-independent set.

Theorem 10. *If c_1, c_2 are different hyperspheres, then $\{c_1, c_2\}$ is an $\perp$-independent set.*

PROOF. By Theorem 9 $\{c_1\}$ is an $\perp$-independent set, and there is an $(n+1)$-simplexoid $\mathfrak{A}=(c_1, a_2, \ldots, a_{n+2})$ with the hypersphere c_1 as one element. Let $\mathfrak{B}=d(\mathfrak{A})$, $\mathfrak{B}=(b_1, b_2, \ldots, b_{n+2})$. By Definition 5 and S4 c_1 is the unique hypersphere such that $c_1\perp\{b_2, \ldots, b_{n+2}\}$, and because of $c_1\neq c_2$, there is at least one hypersphere in the set $\{b_2, \ldots, b_{n+2}\}$ which is not orthogonal to c_2. By symmetry we may assume that $\neg(c_2\perp b_2)$, and then by Theorem 8 it follows that (c_1, c_2, a_3, a_{n+2}) is an $(n+1)$-simplexoid, and therefore $\{c_1, c_2\}$ is an $\perp$-independent set.

Theorem 11. *Let $m\in\{2, \ldots, n+2\}$. If $\{c_1, \ldots, c_{m-1}\}\subseteq S$ in an $\perp$-independent set and c_m, d_m such hyperspheres that*

$$(\{c_1, \ldots, c_{m-1}\}\perp d_m)\,\&\,\neg\,(c_m\perp d_m),$$

then $\{c_1, \ldots, c_{m-1}, c_m\}$ is an $\perp$-independent set.

PROOF. By the hypothesis there is an $(n+1)$-simplexoid $\mathfrak{A}=(c_1, \ldots, c_{m-1}, a_m, \ldots, a_{n+2})$. Let $\mathfrak{B}=d(\mathfrak{A})$, $\mathfrak{B}=(b_1, b_2, \ldots, b_{n+2})$. By Theorem 6 it follows that there is no hypersphere which is orthogonal to the set $A=\{c_1, \ldots, c_{m-1}, a_m, \ldots, a_{n+2}\}$. Hence there is at least one hypersphere in A which is not orthogonal to d_m. But $\{c_1, \ldots, c_{m-1}\}\perp d_m$, and by symmetry we can suppose that $\neg(a_m\perp d_m)$. Then by Theorem 8 $\mathfrak{D}=(b_1, \ldots, b_{m-1}, d_m, b_{m+1}, \ldots, b_{n+2})$ is an $(n+1)$-simplexoid. Let $\mathfrak{E}=d(\mathfrak{D})$, $\mathfrak{E}=(e_1, e_2, \ldots, e_{n+2})$ and $\alpha\in\{1, \ldots, m-1\}$. By Definition 1 it follows for the $(n+1)$-simplexoides $\mathfrak{D}$, $\mathfrak{E}$ that $e_\alpha\perp b_\beta$ for $\forall\beta\in\{1, 2, \ldots, n+2\}\setminus\{m, \alpha\}$ and $e_\alpha\perp d_m$. By the hypothesis it is $c_\alpha\perp d_m$, and for the $(n+1)$-simplexoides $\mathfrak{A}$, $\mathfrak{B}$ we have from Definition 1 $c_\alpha\perp b_\beta$ for $\forall\beta\in\{1, 2, \ldots, n+2\}\setminus\{m, \alpha\}$. Hence

$$\{c_\alpha, e_\alpha\}\perp\{b_1, \ldots, b_{\alpha-1}, b_{\alpha+1}, \ldots, b_{m-1}, d_m, b_{m+1}, \ldots, b_{n+2}\}.$$

But

$$\{b_1, \ldots, b_{\alpha-1}, b_{\alpha+1}, \ldots, b_{m-1}, d_m, b_{m+1}, \ldots, b_{n+2}\}$$

is an $\perp$-independent set, and by S4 it follows $e_\alpha=c_\alpha$ for $\forall\alpha\in\{1, \ldots, m-1\}$, i.e. $\mathfrak{E}=(c_1, \ldots, c_{m-1}, e_m, \ldots, e_{n+2})$. By the hypothesis we have $\neg(c_m\perp d_m)$ and by Theorem 8 it follows that $(c_1, \ldots, c_{m-1}, c_m, e_{m+1}, \ldots, e_{n+2})$ is an $(n+1)$-simplexoid. Hence $\{c_1, \ldots, c_{m-1}, c_m\}$ is an $\perp$-independent set.

Theorem 12. *Let $m\in\{2, \ldots, n+2\}$. If $c_1, c_2, \ldots, c_m, d_2, \ldots, d_m$ are hyperspheres such that $c_\alpha\perp d_\beta$ for $\forall\alpha\in\{1, 2, \ldots, m\}$, $\forall\beta\in\{\alpha+1, \ldots, m\}$ and $\neg(c_\alpha\perp d_\alpha)$ for $\forall\alpha\in\in\{2, \ldots, m\}$, then $\{c_1, c_2, \ldots, c_m\}$ is an $\perp$-independent set.*

PROOF. The theorem can be proved by induction. For $m=2$ the statement follows by Theorems 9 and 11. If $\{c_1, \dots, c_{m-1}\}$ is an $\perp$-independent set, then by Theorem 11 $\{c_1, \dots, c_{m-1}, c_m\}$ is an $\perp$-independent set too.

Theorem 13. *Let* $m \in \{1, 2, \dots, n+1\}$. *If* $\mathfrak{A}=(a_1, a_2, \dots, a_{n+2})$, $\mathfrak{B}=(b_1, b_2, \dots, \dots, b_{n+2})$ *are dual* $(n+1)$*-simplexoides and* c, d *hyperspheres such that* $c \perp \{b_{m+1}, \dots, \dots, b_{n+2}\}$ *and* $\{a_1, \dots, a_m\} \perp d$, *then* $c \perp d$.

PROOF. If $m=1$, then $a_1 \perp d$ and $c \perp \{b_2, \dots, b_{n+2}\}$. By Definition 1 and S4 it follows $c=a_1$, hence $c \perp d$. If $m=n+1$, then $c \perp b_{n+2}$ and $\{a_1, \dots, a_{n+1}\} \perp d$. By Definition 1 and S4 it follows $d=b_{n+2}$, hence $c \perp d$. Let now $m \in \{2, \dots, n\}$. Then $c \perp \{b_{m+1}, \dots, \dots, b_{n+2}\}$ and $\{a_1, \dots, a_m\} \perp d$. By Definition 5 it follows that $a_\alpha \perp b_\beta$ for $\forall \alpha \in \{1, 2, \dots, \dots, n+1\}$ and $\forall \beta \in \{\alpha+1, \dots, n+1\}$, and $(\neg a_\alpha \perp b_\alpha)$ for $\forall \alpha \in \{2, \dots, n+1\}$. Let us suppose, contrary to the statement, that $\neg (c \perp d)$. Put

$$a_1' = a_1, \dots, a_m' = a_m, \quad a_{m+1}' = c, \quad a_{m+2}' = a_{m+1}, \dots, a_{n+2}' = a_{n+1},$$

$$b_1' = b_1, \dots, b_m' = b_m, \quad b_{m+1}' = d, \quad b_{m+2}' = b_{m+1}, \dots, b_{n+2}' = b_{n+1}.$$

Then $a_\alpha' \perp b_\beta'$ for $\forall \alpha \in \{1, 2, \dots, n+2\}$, $\forall \beta \in \{\alpha+1, \dots, n+2\}$ and $\neg(a_\alpha' \perp b_\alpha')$ for $\forall \alpha \in \in \{2, \dots, n+2\}$. From Theorem 12 for $m=n+2$ it follows that $\{a_1', a_2', \dots, a_{n+2}'\} = =\{a_1, \dots, a_m, c, a_{m+1}, \dots, a_{n+1}\}$ is an $\perp$-independent set. Then by Theorem 6 it follows that there is no hypersphere which is orthogonal to the set $\{a_1, \dots, a_m, c, a_{m+1}, \dots, a_{n+1}\}$. But by Definition 5 it follows $\{a_1, \dots, a_m, a_{m+1}, \dots, a_{n+1}\} \perp b_{n+2}$, and by the hypothesis $c \perp b_{n+2}$. Hence $\{a_1, \dots, a_m, c, a_{m+1}, \dots, a_{n+1}\} \perp b_{n+2}$. This contradiction proves the statement of the theorem.

4. Subspaces

Definition 6. If P, Q are nonempty sets of hyperspheres, then we say that P is saturated with Q or that Q saturates P iff:

M1. $P \perp Q$,

M2. $(\forall s \in S)\, [s \perp Q \Rightarrow s \in P]$.

An equivalent form of Definition 6 is obviously

Theorem 14. *If* P, Q *are nonempty sets of hyperspheres, then* P *is saturated with* Q *iff* P *is the set of all hyperspheres* p *such that* $p \perp Q$.

We shall say, by convention, that the set S of all hyperspheres is saturated with the empty set $\emptyset$, and conversely that $\emptyset$ is saturated with S, since here the conditions M1 and M2 are satisfied in a trivial way.

Definition 7. A set P of hyperspheres will be called the subspace of the space $(S, \perp)$ iff there is a set Q of hyperspheres such that P is saturated with Q.

Definition 8. Two subspaces P, Q of the space $(S, \perp)$ will be called complementary subspaces, and it will be written $P=C(Q)$ or $Q=C(P)$, iff P is saturated with Q and Q is saturated with P.

Obviously $S=C(\emptyset)$, $\emptyset=C(S)$.

From Definitions 6 and 8 it follows at once

Theorem 15. *The sets P, Q of hyperspheres are complementary subspaces of the space $(S, \perp)$ iff:*

M1. $P \perp Q$,

M2. $(\forall s \in S)[s \perp Q \Rightarrow s \in P]$,

M3. $(\forall s \in S)[s \perp P \Rightarrow s \in Q]$.

Analogously, from Theorem 14 and Definition 8 we get

Theorem 16. *The sets P, Q of hyperspheres are complementary subspaces of the space $(S, \perp)$ iff:*

a) $(\forall s \in S)[s \perp Q \Leftrightarrow s \in P]$,

b) $(\forall s \in S)[s \perp P \Leftrightarrow s \in Q]$.

It is obvious that the set P in Definition 6 is uniquely determined by the set Q (the converse is not generally valid), and hence each subspace of the space $(S, \perp)$ is uniquely determined by its complementary subspace.

Theorem 17. *If P_1, Q_1 and P_2, Q_2 are two pairs of complementary subspaces of the space $(S, \perp)$, then the relations $P_1 \subset P_2$ and $Q_1 \supset Q_2$ are equivalent.*

Proof. It is obvious that from $P_1 \subset P_2$ follows $Q_1 \supseteq Q_2$. If it were $Q_1 = Q_2$ then it would follow $P_1 = P_2$. Hence $Q_1 \supset Q_2$. The converse is proved analogously.

Theorem 18. *If the set P of hyperspheres is saturated with the set Q of hyperspheres and Q is saturated with the set P_0 of hyperspheres then P, Q are complementary subspaces of the space $(S, \perp)$.*

Proof. Since Q is saturated with P_0 and P is saturated with Q, it follows by Definition 6

$$P_0 \perp Q, \tag{6}$$

$$(\forall s \in S)[P_0 \perp s \Rightarrow s \in Q], \tag{7}$$

$$P \perp Q, \tag{8}$$

$$(\forall s \in S)[s \perp Q \Rightarrow s \in P]. \tag{9}$$

Let $s \in P_0$. Then by (6) $s \perp Q$ and by (9) we get $s \in P$. Hence $P_0 \subseteq P$. Now, let $s \in S$. If $P \perp s$, then $P_0 \perp s$ and by (7) $s \in Q$. Hence

$$(\forall s \in S)[P \perp s \Rightarrow s \in Q]. \tag{10}$$

From (8), (9) and (10) we get by Theorem 15 that P and Q are complementary subspaces of the space $(S, \perp)$.

5. Basic sets and dimension of a subspace

Definition 9. We say that a subspace P of the space $(S, \perp)$ is generated by the set P_0 of hyperspheres (or that the set P_0 is the set of generators of the subspace P), and write $P=\langle P_0\rangle$, iff there is a set Q of hyperspheres such that P is saturated with Q and Q is saturated with P_0.

From the proof of Theorem 18 it follows that $\langle P_0\rangle=P$ implies $P_0\subseteqq P$.

Theorem 19. *If P_0 is a nonempty set of hyperspheres, then there is a set $\bar{P}_0\subseteqq P_0$ of hyperspheres such that $\bar{P}_0$ is an $\perp$-independent set and $\langle\bar{P}_0\rangle=\langle P_0\rangle$.*

Proof. There is a greatest number m, $m\in\{-1, 0, 1, \dots, n\}$, such that there exists an $(n+1)$-simplexoid with $m+2$ elements from the set P_0. Let $\mathfrak{P}=(p_1, p_2, \dots, p_{n+2})$ be such an $(n+1)$-simplexoid and let $p_1, \dots, p_{m+2}$ be its elements from the set P_0. Put $\bar{P}_0=\{p_1, \dots, p_{m+2}\}$. Let Q, $\bar{Q}$ be the sets of hyperspheres saturated with P_0, $\bar{P}_0$ and P, $\bar{P}$ the sets of hyperspheres saturated with Q, $\bar{Q}$, respectively. Then, by Definition 9 it follows $P=\langle P_0\rangle$, $\bar{P}=\langle\bar{P}_0\rangle$. It is necessary to prove that $P=\bar{P}$. By Theorem 18 P, Q resp. $\bar{P}$, $\bar{Q}$ are complementary subspaces of the space $(S, \perp)$ and it is sufficient to prove $Q=\bar{Q}$. If $q\in Q$, then $P_0\perp q$ and by $\bar{P}_0\subseteqq P_0$ it follows $\bar{P}_0\perp q$, hence $q\in\bar{Q}$. Therefore we have $Q\subseteqq\bar{Q}$. We prove $\bar{Q}\subseteqq Q$. There are two cases:

1) $m=n$. The elements of the set $\bar{P}_0=\{p_1, p_2, \dots, p_{n+2}\}$ constitute an $(n+1)$-simplexoid $\mathfrak{P}=(p_1, p_2, \dots, p_{n+2})$ and by Theorem 6 it follows $\bar{Q}=\emptyset$, wherefrom by $Q\subseteqq\bar{Q}$ we get $Q=\emptyset$, and so $Q=\bar{Q}$.

2) $m\in\{-1, 0, 1, \dots, n-1\}$. Let $q\in\bar{Q}$. Then $\bar{P}_0\perp q$, i.e. $\{p_1, \dots, p_{m+2}\}\perp q$. If it were $q\notin\bar{Q}$, a hypersphere $p'\in P_0$ would exist such that $\neg(p'\perp q)$ and then by Theorem 11 the subset $\{p_1, \dots, p_{m+2}, p'\}\subseteqq P_0$ would be an $\perp$-independent set, which contradicts the definition of the number m. Therefore $q\in Q$, and $\bar{Q}\subseteqq Q$.

Definition 10. The set of generators P of the subspace P of the space $(S, \perp)$ will be called the basic set of that subspace iff P is an $\perp$-independent set.

Theorem 20. *If P, Q are nonempty complementary subspaces of the space $(S, \perp)$, then there is at least one pair of dual $(n+1)$-simplexoides $\mathfrak{P}=(p_1, p_2, \dots, p_{n+2})$, $\mathfrak{Q}=(q_1, q_2, \dots, q_{n+2})$ of hyperspheres and a uniquely determined number $m\in\{-1, 0, 1, \dots, n-1\}$, such that $P=\langle\{p_1, \dots, p_{m+2}\}\rangle$, $Q=\langle\{q_{m+3}, \dots, q_{n+2}\}\rangle$.*

Proof. Obviously $\langle P\rangle=P$. By Theorem 19 there is a set $\bar{P}\subseteqq P$ of hyperspheres such that $\langle\bar{P}\rangle=P$ and $\bar{P}$ is an $\perp$-independent set. Let $p_1, \dots, p_{m+2}$, $m\in\{-1, 0, 1, \dots, n-1\}$ be the elements of the set P. Then these hyperspheres are elements of an $(n+1)$-simplexoid $\mathfrak{P}=(p_1, p_2, \dots, p_{n+2})$. Let $\mathfrak{Q}=d(\mathfrak{P})$, $\mathfrak{Q}=(q_1, q_2, \dots, q_{n+2})$. The set Q consists of all those hyperspheres that are orthogonal to the set P and by Definition 5 it follows $q_{m+3}, \dots, q_{n+2}\in Q$. Let $\bar{Q}=\{q_{m+3}, \dots, q_{n+2}\}$. Owing to $\bar{Q}\subseteqq Q$,

$$p\perp Q \tag{11}$$

is valid for $\forall p\in P$. Conversely, if for some $p\in S$ (11) is valid, then by Theorem 13 it follows $p\perp q$ for all hyperspheres q for which

$$\bar{P}\perp q. \tag{12}$$

But, since $\bar{P}$ is a set of generators of P, from (12) if follows $q \in Q$ and hence $p \perp q$ for $\forall q \in Q$. By Definition 6 $p \in P$. Therefore $\bar{Q}$ is a set of generators of Q. Suppose now that there are two pairs, identical or different, of dual $(n+1)$-simplexoides of hyperspheres

$$\mathfrak{P} = (p_1, p_2, \ldots, p_{n+2}), \quad \mathfrak{Q} = (q_1, q_2, \ldots, q_{n+2})$$

and

$$\mathfrak{P}' = (p_1', p_2', \ldots, p_{n+2}'), \quad \mathfrak{Q}' = (q_1', q_2', \ldots, q_{n+2}')$$

and numbers $m, m' \in \{-1, 0, 1, \ldots, n-1\}$ such that

$$\bar{P} = \{p_1, \ldots, p_{m+2}\}, \quad \bar{P}' = \{p_1', \ldots, p_{m'+2}'\}$$

are the sets of generators of P and

$$\bar{Q} = \{q_{m+3}, \ldots, q_{n+2}\}, \quad \bar{Q}' = \{q_{m'+3}', \ldots, q_{n+2}'\}$$

the sets of generators of Q. We show that $m=m'$. Suppose e.g. that $m<m'$. Then $m \in \{-1, 0, 1, \ldots, n-2\}$. Therefore there are no more than $n+1$ hyperspheres among the hyperspheres

$$(13) \qquad p_1, \ldots, p_{m+2}, p_{m'+3}', \ldots, p_{n+2}'$$

and by S2 there is a hypersphere q which is orthogonal to each of the hyperspheres (13). The hypersphere q is orthogonal to the set $\bar{P}$, and as $\langle \bar{P} \rangle = P$, we get $q \in Q$. On the other hand from $\langle \bar{P}' \rangle = P$ it follows $\bar{P}' \subseteq P$ and hence $\bar{P}' \perp q$. Therefore $\{p_1', \ldots, p_{m'+2}', p_{m'+3}', \ldots, p_{n+2}'\} \perp q$, which contradicts Theorem 6. According to this it cannot be $m<m'$, and by symmetry it cannot be $m'<m$ either. Thus $m=m'$.

By Theorem 20 and Definition 10 it follows that each nonempty subspace of the space $(S, \perp)$ has a basic set and that all basic sets have the same number of hyperspheres which is precisely $m+2$, $m \in \{-1, 0, 1, \ldots, n\}$. For the basic set of the subspace S we can take each $\perp$-independent subset of $n+2$ hyperspheres. This property enables us to give the following definition:

Definition 11. The number of elements of any basic set of a nonempty subspace P of the space $(S, \perp)$ diminished by 2 is called the dimension of P with respect to the relation "$\perp$" or the $\perp$-dimension of P. If the $\perp$-dimension of P is m, then we write $\dim_\perp P = m$. In particular we define $\dim_\perp \emptyset = -2$.

Obviously $\dim_\perp S = n$.

By Theorem 20 and Definitions 10 and 11 it follows immediately

Theorem 21. *If P and Q are complementary subspaces of the space $(S, \perp)$, then* $\dim_\perp P + \dim_\perp Q = n-2$.

Theorem 22. *If P_1 and P_2 are subspaces of the space $(S, \perp)$ then from $P_1 \subset P_2$ it follows* $\dim_\perp P_1 < \dim_\perp P_2$.

PROOF. Obviously from $P_1 \subset P_2$ it follows $\dim_\perp P_1 \leqq \dim_\perp P_2$. Suppose that $\dim_\perp P_1 = \dim_\perp P_2$. Then the basic sets of the subspaces P_1 and P_2 have the same number of elements, and each basic set of P_1 is simultaneously the basic set of P_2. As the subspace is uniquely determined by each of its sets of generators, it follows $P_1 = P_2$ in contradiction with the hypothesis $P_1 \subset P_2$. Therefore $\dim_\perp P_1 < \dim_\perp P_2$.

Theorem 23. *If P is a subspace of the space $(S, \perp)$, then the relation* $\dim_\perp P = -1$ *is valid iff the set P has exactly one element. For $\forall p_1 \in S$ the set $\{p_1\}$ is the subspace of the space $(S, \perp)$.*

Proof. Let P be a subspace of the space $(S, \perp)$. If P contains exactly one element then by Theorem 9 it follows that $\dim_\perp P = -1$. Conversely, let $\dim_\perp P = -1$. If then subspace P would contain two different hyperpherses then by Theorem 10 it would be $\dim_\perp P \geqq 0$ which is impossible. On the other hand from $P = \emptyset$ it would follow by Definition 11 that $\dim_\perp P = -2$. Therefore the subspace P contains exactly one hypersphere. Let now p_1 be any hypersphere, and let $P = \langle\{p_1\}\rangle$. Then $\dim_\perp P \geqq -1$. Let $Q = C(P)$. By Theorem 9 there is an $(n+1)$-simplexoid $\mathfrak{P} = (p_1, p_2, \ldots, \ldots, p_{n+2})$ with the hypersphere p_1 as one element. Let $\mathfrak{Q} = d(\mathfrak{P})$, $\mathfrak{Q} = (q_1, q_2, \ldots, q_{n+2})$. By Definition 5 $p_1 \perp \{q_2, \ldots, q_{n+2}\}$, and since the set Q is saturated (Theorem 18) with the set $\{p_1\}$, it follows that $q_2, \ldots, q_{n+2} \in Q$. Since $\{q_2, \ldots, q_{n+2}\}$ is an $\perp$-independent set, $\dim_\perp Q \geqq n-1$, wherefrom by Theorem 21 it follows that $\dim_\perp P \leqq -1$. Therefore, together with $\dim_\perp P \geqq -1$, we get $\dim_\perp P = -1$, and by the first part of the theorem it follows $P = \{p_1\}$.

6. Intersection and sum of subspaces

Theorem 24. *If P_1, Q_1 and P_2, Q_2 are two pairs of complementary subspaces of the space $(S, \perp)$ then $P_1 \cap P_2$ is a subspace too. If $Q = C(P_1 \cap P_2)$ then $Q = \langle Q_1 \cup Q_2 \rangle$.*

Proof. Let Q be the set of hyperspheres saturated with the set $P_1 \cap P_2$. As the sets P_1, P_2, Q are saturated with the sets $Q_1, Q_2, P_1 \cap P_2$ respectively, so by Definition 6 it follows that

$$P_1 \perp Q_1, \tag{14}$$

$$(\forall s \in S)[s \perp Q_1 \Rightarrow s \in P_1], \tag{15}$$

$$P_2 \perp Q_2, \tag{16}$$

$$(\forall s \in S)[s \perp Q_2 \Rightarrow s \in P_2], \tag{17}$$

$$(P_1 \cap P_2) \perp Q, \tag{18}$$

$$(\forall s \in S)[(P_1 \cap P_2) \perp s \Rightarrow s \in Q]. \tag{19}$$

Let $s \in P_1 \cap P_2$. Then from (14) and (16) follows $s \perp Q_1$, $s \perp Q_2$. Hence $s \perp (Q_1 \cup Q_2)$. Therefore we have

$$(P_1 \cap P_2) \perp (Q_1 \cup Q_2). \tag{20}$$

Let now $s \in S$. If $s \perp (Q_1 \cup Q_2)$, then $s \perp Q_1$, $s \perp Q_2$, and by (15) and (17) it follows $s \in P_1$, $s \in P_2$, i.e. $s \in P_1 \cap P_2$. Therefore we have

$$(\forall s \in S)[s \perp (Q_1 \cup Q_2) \Rightarrow s \in P_1 \cap P_2]. \tag{21}$$

From (20) and (21) it follows by Definition 6 that $P_1 \cap P_2$ is saturated with the set $Q_1 \cup Q_2$. Thus by Definition 9 $\langle Q_1 \cup Q_2 \rangle = Q$ and hence $Q \supseteq Q_1 \cup Q_2$. Let now $s \in S$. If $s \perp Q$ then $s \perp (Q_1 \cup Q_2)$ and by (21) we get $s \in P_1 \cap P_2$. Therefore

$$(\forall s \in S)[s \perp Q \Rightarrow s \in P_1 \cap P_2]. \tag{22}$$

From (18), (19) and (22) it follows by Theorem 15 that $P_1 \cap P_2$, Q are complementary subspaces of the space $(S, \perp)$.

Definition 12. The subspace Q of the space $(S, \perp)$, which is complementary to the intersection $P_1 \cap P_2$ of two subspaces P_1, P_2 with $P_1 = C(Q_1)$, $P_2 = C(Q_2)$, will be called the sum of the subspaces Q_1, Q_2, and will be written $Q = Q_1 + Q_2$.

From Theorem 24 and Definition 12 it follows $Q_1 + Q_2 = \langle Q_1 \cup Q_2 \rangle$.

Theorem 25. *If P_1, P_2 are two subspaces of the space $(S, \perp)$, then*

$$C(P_1 \cap P_2) = C(P_1) + C(P_2), \quad C(P_1 + P_2) = C(P_1) \cap C(P_2).$$

PROOF. Let $Q_1 = C(P_1)$, $Q_2 = C(P_2)$. By Definition 12 $P_1 \cap P_2$, $Q_1 + Q_2$ and $P_1 + P_2$, $Q_1 \cap Q_2$ are two pairs of complementary subspaces of the space $(S, \perp)$. Therefore

$$C(P_1 \cap P_2) = Q_1 + Q_2 = C(P_1) + C(P_2), \quad C(P_1 + P_2) = Q_1 \cap Q_2 = C(P_1) \cap C(P_2).$$

Theorem 26. *If P is a subspace of the space $(S, \perp)$ with $\dim_\perp P = m$, $m \in \{-1, 0, 1, \ldots, n-1\}$, and if $\{p_1, \ldots, p_k\} \subseteq P$, $k \in \{1, \ldots, m+1\}$ is an $\perp$-independent set, then there are hyperspheres $p_{k+1}, \ldots, p_{m+2} \in P$ such that $\{p_1, \ldots, p_k, p_{k+1}, \ldots, p_{m+2}\}$ is a basic set of the subspace P.*

PROOF. The greatest number of elements of any $\perp$-independent subset of the subspace P equals $\dim_\perp P + 2 = m + 2$. As $k < m+2$, there is a hypersphere p_{k+1} in the set $P \setminus \{p_1, \ldots, p_k\}$, such that $\{p_1, \ldots, p_k, p_{k+1}\}$ is an $\perp$-independent set. The statement of the theorem follows inductively.

Theorem 27. *If P_1, P_2 are two subspaces of the space $(S, \perp)$ then*

$$\dim_\perp (P_1 \cap P_2) + \dim_\perp (P_1 + P_2) = \dim_\perp P_1 + \dim_\perp P_2. \tag{23}$$

PROOF. Let $\dim_\perp P_1 = i$, $\dim_\perp P_2 = j$, $\dim_\perp (P_1 \cap P_2) = k$. Then obviously $k \leqq i$, $k \leqq j$. The greatest number of elements of any $\perp$-independent subset of the subspaces $P_1, P_2, P_1 \cap P_2$ is $i+2, j+2, k+2$, respectively. First we prove the inequality

$$\dim_\perp (P_1 + P_2) \leqq i + j - k. \tag{24}$$

We distinguish two cases:

1) $k \in \{-1, 0, 1, \ldots, n\}$. Let $\{p_1, \ldots, p_{k+2}\}$ be a basic set of the subspace $P_1 \cap P_2$. By Theorem 26 there are hyperspheres $p_{k+3}, \ldots, p_{i+2} \in P_1$ such that $\{p_1, \ldots, p_{i+2}\}$ is a basic set of the subspace P_1, which in case $k = i$ is reduced to $\{p_1, \ldots, p_{k+2}\}$. Similarly, there are hyperspheres $p_{i+3}, \ldots, p_{i+j-k+2} \in P$ such that $\{p_1, \ldots, p_{k+2}, p_{i+3}, \ldots, \ldots, p_{i+j-k+2}\}$ is a basic set of the subspace P_2, which in case $k = j$ is reduced to $\{p_1, \ldots, \ldots, p_{k+2}\}$. If $Q_1 = C(P_1)$, $Q_2 = C(P_2)$ then by Definition 12 $Q_1 \cap Q_2 = C(P_1 + P_2)$. Since $\{p_1, \ldots, p_{k+2}, p_{k+3}, \ldots, p_{i+2}\}$ is the basic set of the subspace P_1, by Theorem 18 and Definition 9 it follows that for $\forall s \in S$ the relation $\{p_1, \ldots, p_{k+2}, p_{k+3}, \ldots, p_{i+2}\} \perp s$

is valid iff $s \in Q_1$. Similarly the relation $\{p_1, \dots, p_{k+2}, p_{i+3}, \dots, p_{i+j-k+2}\} \perp s$ is equivalent with $s \in Q_2$. Therefore for $\forall s \in S$ the relation

$$\{p_1, \dots, p_{k+2}, p_{k+3}, \dots, p_{i+2}, p_{i+3}, \dots, p_{i+j-k+2}\} \perp s$$

is equivalent with $s \in Q_1 \cap Q_2$, and by Theorem 14 the set $Q_1 \cap Q_2$ is saturated with the set

$$\{p_1, \dots, p_{k+2}, p_{k+3}, \dots, p_{i+2}, p_{i+3}, \dots, p_{i+j-k+2}\}.$$

Since on the other hand the set P_1+P_2 is saturated with the set $Q_1 \cap Q_2$, by Definition 9 it follows that $\{p_1, \dots, p_{i+j-k+2}\}$ is the set of generators of the subspace P_1+P_2. Since the $\perp$-dimension of the subspace P_1+P_2 is equal to the greatest number of the elements of any $\perp$-independent subset of the set $\{p_1, \dots, p_{i+j-k+2}\}$ diminished by 2, relation (24) follows at once.

2) $k=-2$. Now $P_1 \cap P_2 = \emptyset$. If $\{p_1, \dots, p_{i+2}\}$ is a basic set of the subspace P_1 and $\{p_{i+3}, \dots, p_{i+j+4}\}$ is a basic set of the subspace P_2, then similarly as in the case 1) we prove that $\{p_1, \dots, p_{i+2}, p_{i+3}, \dots, p_{i+j+4}\}$ is a set of generators of the subspace P_1+P_2, and hence $\dim_\perp (P_1+P_2) \leqq i+j+2$, wherefrom by $k=-2$ the relation (24) again follows.

Thus in both cases the relation (24) holds. By Definition 12 $Q_1+Q_2 = C(P_1 \cap P_2)$, and by the use of Theorem 21 we get

$$\dim_\perp (Q_1 \cap Q_2) = n-2-\dim_\perp (P_1+P_2), \tag{25}$$

$$\dim_\perp (Q_1+Q_2) = n-2-\dim_\perp (P_1 \cap P_2), \tag{26}$$

$$\dim_\perp Q_1 = n-2-\dim_\perp P_1, \quad \dim_\perp Q_2 = n-2-\dim_\perp P_2. \tag{27}$$

From (24) by the definition of the numbers i, j, k it follows

$$\dim_\perp (P_1 \cap P_2) + \dim_\perp (P_1+P_2) \leqq \dim_\perp P_1 + \dim_\perp P_2. \tag{28}$$

Similarly

$$\dim_\perp (Q_1 \cap Q_2) + \dim_\perp (Q_1+Q_2) \leqq \dim_\perp Q_1 + \dim_\perp Q_2, \tag{29}$$

and hence by (25), (26) and (27) it follows

$$\dim_\perp (P_1+P_2) + \dim_\perp (P_1 \cap P_2) \geqq \dim_\perp P_1 + \dim_\perp P_2. \tag{29}$$

From (28) and (29) we get just the required relation (23).

Theorem 28. *If P and Q are subspaces of the space $(S, \perp)$ then*

$$\dim_\perp [P \cap C(Q)] = \dim_\perp P - \dim_\perp Q + \dim_\perp [C(P) \cap Q].$$

Proof. By Theorem 21 $\dim_\perp [P \cap C(Q)] = n-2-\dim_\perp C[P \cap C(Q)]$, and as by Theorem 25 $C[P \cap C(Q)] = C(P) + C[C(Q)] = C(P)+Q$, so $\dim_\perp [P \cap C(Q)] = n-2-\dim_\perp [C(P)+Q]$. By Theorem 27 $\dim_\perp [C(P)+Q] = \dim_\perp C(P) + \dim_\perp Q - \dim_\perp [C(P) \cap Q]$, and hence $\dim_\perp [P \cap C(Q)] = n-2-\dim_\perp C(P) - \dim_\perp Q + \dim_\perp [C(P) \cap Q]$. By Theorem 21 $n-2-\dim_\perp C(P) = \dim_\perp P$, and therefore the required equation follows.

7. Induced Möbius geometries

Definition 13. A hypersphere s will be called a point-hypersphere or shortly a point iff $s \perp s$.

Definition 14. The subspace P of the space $(S, \perp)$ will be called regular iff $P \cap C(P) = \emptyset$ and singular otherwise.

Theorem 29. *If a subspace P of the space $(S, \perp)$ contains no point P is a regular subspace.*

PROOF. Suppose that P is a singular subspace. Then by Definition 14 there is a hypersphere $p \in P \cap C(P)$ and by the property M1 it follows $p \perp p$, i.e. p is a point, which is in contradiction with the hypothesis.

Theorem 30. *If P is a regular subspace of the space $(S, \perp)$ and P_0 is a subspace of $(S, \perp)$ such that $P_0 \subseteq P$, then* $\dim_\perp [P \cap C(P_0)] = \dim_\perp P - \dim_\perp P_0 - 2$.

PROOF. By Definition 14 $P \cap C(P) = \emptyset$ and hence $P_0 \cap C(P) = \emptyset$. Since $\dim_\perp \emptyset = -2$, the statement follows immediately from Theorem 28.

Theorem 31. *If P is a regular subspace of the space $(S, \perp)$ with $\dim_\perp P = m$, $m \in \{0, 1, \ldots, n-1\}$, and $\{p_1, \ldots, p_k\}$, $k \in \{1, \ldots, m+2\}$ is an $\perp$-independent subset of P, then there is an $(m+1)$-simplexoid with elements from P, so that $p_1, \ldots, p_k$ are the elements of that $(m+1)$=simpleoxid.*

PROOF. If $k < m+2$ then by Theorem 26 there are hyperspheres $p_{k+1}, \ldots, p_{m+2} \in P$ such that $\{p_1, \ldots, p_{m+2}\}$ is a basic set of the subspace P. For $k = m+2$ this statement is trivially realized. Let be now $P_\alpha = \langle\{p_1, \ldots, p_{\alpha-1}, p_{\alpha+1}, \ldots, p_{m+2}\}\rangle$ for $\forall \alpha \in \{1, \ldots, m+2\}$. Then $P_\alpha \subset P$ and $\dim_\perp P_\alpha = m-1$. Hence by Theorem 30 $\dim_\perp [P \cap C(P_\alpha)] = -1$, and by Theorem 23 it follows that the subspace $P \cap C(P_\alpha)$ contains exactly one hypersphere. Denote this hypersphere with q_α. For $\forall \alpha \in \{1, \ldots, m+2\}$ we have $q_\alpha \in P \cap C(P_\alpha)$ and hence $q_\alpha \in C(P_\alpha)$. By the property M1 it follows $\{p_1, \ldots, p_{\alpha-1}, p_{\alpha+1}, \ldots, p_{m+2}\} \perp q_\alpha$. Suppose now that $p_\alpha \perp q_\alpha$. Then $\{p_1, \ldots, p_{m+2}\} \perp q_\alpha$. Since $\{p_1, \ldots, p_{m+2}\}$ is the basic set of the subspace P, $q_\alpha \in C(P)$. From $q_\alpha \in P$ and $q_\alpha \in C(P)$ by Definition 14 we get that P is a singular subspace, contrary to hypothesis. Therefore

$$(\forall \alpha, \beta \in \{1, \ldots, m+2\})[p_\alpha \perp q_\beta \Leftrightarrow \alpha \neq \beta],$$

i.e. $(p_1, \ldots, p_{m+2})$ is an $(m+1)$-simplexoid with the elements from P.

Definition 15. If P is a subset of S and "$\top$" a binary relation on P such that

$$(\forall p, q \in P)[p \top q \Leftrightarrow p \perp q],$$

then we say that "$\top$" is the relation on P induced by the relation of orthogonality.

From Theorem 31 and Definitions 1, 2 and 15 it follows immediately

Theorem 32. If P is a regular subspace of the space $(S, \perp)$ and "$\top$" is the relation on P induced by the relation of orthogonality, then every $\perp$-independent subset of P is a $\top$-independent set.

The converse of Theorem 32 is also true, even under weaker conditions.

Theorem 33. If P is a subspace of the space $(S, \perp)$ and "$\top$" the relation on P induced by the relation of orthogonality, then every $\top$-independent subset of P is an $\perp$-independent set.

Proof. Let $\{p_1, \dots, p_k\}$, $k \in \{1, \dots, m+2\}$ be a $\top$-independent subset of P. Then by Definition 2 there is an $(m+1)$-simplexoid $\mathfrak{P}=(p_1, \dots, p_{m+2})$ with elements $p_1, \dots, p_k$ and moreover some elements $p_{k+1}, \dots, p_{m+2} \in P$. Let $\mathfrak{Q}=d(\mathfrak{P})$, $\mathfrak{Q}=(q_1, \dots, q_{m+2})$. The statement of the theorem will be proved if we prove that $P_0=\{p_1, \dots, p_{m+2}\}$ is an $\perp$-independent set. Suppose, to the contrary, that P_0 is not an $\perp$-independent set. By Theorem 19 there is an $\perp$-independent subset $\bar{P}_0$ of P_0 such that the sets P_0 and $\bar{P}_0$ generate the same subspace Q of the space $(S, \perp)$. By symmetry we can suppose that $\bar{P}_0=\{p_2, \dots, p_j\}$, $j \in \{2, \dots, m+2\}$. Since by Definition 1 $P_0 \perp q_1$ and hence $\bar{P}_0 \perp q_1$, and since $\bar{P}_0$ is a basic set of the subspace Q of $(S, \perp)$, hence $q_1 \in C(Q)$. Since on the other hand $p_1 \in P_0 \subseteqq Q$, by the property M1 it follows $p_1 \perp q_1$, i.e. $p_1 \top q_1$, which contradicts Definition 5. This proves our statement.
Now we can prove

Theorem 34. *If P is a regular subspace of the space $(S, \perp)$ with $\dim_\perp P=m$, $m \in \{0, 1, \dots, n-1\}$, and if "$\top$" is the relation on P induced by the relation of orthogonality, then $(P, \top)$ is an m-dimensional Möbius space.*

Proof. The property S1 of the structure $(P, \top)$ follows from the same property of the structure $(S, \perp)$. Let $P_0=\{p_1, \dots, p_{m+1}\}$ be any subset of P. Let further $Q=\langle P_0\rangle$. Then $Q \subset P$ and $\dim_\perp Q \leqq m-1$. Thus by Theorem 30 it follows $\dim_\perp [P \cap C(Q)] \geqq -1$. This by Theorem 23 means that the subspace $P \cap C(Q)$ contains at least one hypersphere q. Then $q \in P$ and $q \in C(Q)$, and by the property M1 it follows $P_0 \perp q$, i.e. $P_0 \top q$. In particular $\{p_1, \dots, p_{m+1}\} \top q$, that proves the property S2 of the structure $(P, \top)$. If P is a $\top$-independent set, then by Theorem 33 P_0 is an $\perp$-independent set and hence $\dim_\perp Q=m-1$. Therefore $\dim_\perp [P \cap C(Q)]=-1$, and by Theorem 23 there is exactly one hypersphere $q \in P \cap C(Q)$, i.e. a hypersphere q such that $\{p_1, \dots, p_{m+1}\} \top q$, which proves the property S4 of the structure $(P, \top)$. Let now $\{p_1, \dots, p_{m+2}\}$ be any basic set of the subspace P. If there were a hypersphere $q \in P$ such that $\{p_1, \dots, p_{m+2}\} \top q$, i.e. $\{p_1, \dots, p_{m+2}\} \perp q$, it would be $q \in C(P)$, and hence $q \in P \cap C(P)$, which is impossible, while P is a regular subspace of the space $(S, \perp)$. Thus the property S3 of the structure $(P, \top)$ is proved, and by Definition 4 it follows that $(P, \top)$ is an m-dimensional Möbius space.

By the definition of the relation "$\top$" and the definition of subspaces it follows immediately that every subspace of the space $(P, \top)$ is also a subspace of the space $(S, \perp)$ and conversely that every subspace of the space $(S, \perp)$ which is a subset of P, is also a subspace of the space $(P, \top)$.

Finally, by Theorems 32 and 33 we get

Theorem 35. *If P is a regular subspace of the space $(S, \perp)$, "$\top$" the relation on P induced by the relation of orthogonality, and Q any subspace of the space $(P, \top)$, and hence also a subspace of the space $(S, \perp)$, then* $\dim_\perp Q=\dim_\top Q$.

Therefore, the structure of the space $(P, \top)$ is identical with the structure of the subspace P induced by the structure of the space $(S, \perp)$.

8. Remark

By the comparison of system of axioms S1—S4 and system of axioms P1—P4 from [7] it follows that an n-dimensional Möbius space is in fact an $(n+1)$-dimensional pre-projective space. (About the pre-projective plane see the articles [1], [2] and [3] of V. DEVIDÉ.)

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