On Moufang and extra loops

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In this paper, indentities which arise fromMoufang and extra loop identities are considered. These lead to characterizations of Moufang and extra loops and groups. In the first section Moufang type identities are treated. The second section contains extra loop type identities. The last section deals with the characterizations.

1. Moufang type

Let $G(\cdot)$ be a loop. Then it is known that, in $G(\cdot)$ the following indentities are equivalent:

$$\text{(i)} \qquad (yx\cdot z)x = y\cdot(x\cdot zx)$$

$$\text{(ii)} \qquad xy\cdot zx = (x\cdot yz)x$$

$$\text{(iii)} \qquad xy\cdot zx = x\cdot(yz\cdot x)$$

$$\text{(iv)} \qquad (xy\cdot x)z = x\cdot(y\cdot xz).$$

A loop $G(\cdot)$ satisfying any one (and hence all) of these identities, is called a Moufang loop [1, p. 115]. Corresponding to the above well known identities, the following identical relations are considered:

$$(1) \qquad (yx\cdot z)\cdot\lambda x = y\cdot\big(x\cdot(z\cdot\lambda x)\big)$$

$$(2) \qquad xy\cdot(z\cdot\lambda x) = (x\cdot yz)\cdot\lambda x$$

$$(3) \qquad xy\cdot(z\cdot\lambda x) = x\cdot(yz\cdot\lambda x)$$

$$(4) \qquad (xy\cdot\lambda x)z = x\cdot\big(y\cdot(\lambda x\cdot z)\big)$$

where $\lambda\colon G\to G$ is any mapping.

We prove the following results regarding (1), (2), (3) and (4). In the sequel we make use of the following fact about inverse property loops $G(\cdot)$: If (U, V, W) is an autotopism of G, then (W, JVJ, U) and (JUJ, W, V), where J is the inverse mapping are also autotopisms of G [1, p. 112].

Theorem 1. *If* (1) *holds in a loop* $G(\cdot)$, *then* $G(\cdot)$ *is Moufang and that* (2), (3) *and* (4) *also hold in* $G(\cdot)$. *Further* $\mu x \in N$ *(nucleus) where* $\mu: G \to G$ *is a mapping such that* $x \cdot \mu x = \lambda x$. *Conversely, if* $G(\cdot)$ *is a Moufang loop with* $\mu x \in N$, *for all* $x \in G$, *then the identity* (1) *and hence* (2), (3) *and* (4) *all hold in* G.

PROOF. Let ${}^{-1}x, x^{-1} \in G$ such that

$$(5) \qquad {}^{-1}x \cdot x = 1 = x \cdot x^{-1}, \quad \text{for all} \quad x \in G.$$

Putting $y = {}^{-1}x$ in (1) and using (5), we get

$$z \cdot \lambda x = {}^{-1}x \cdot (x \cdot (z \cdot \lambda x)),$$

that is,

$$(6) \qquad u = {}^{-1}x \cdot (xu), \quad \text{for all} \quad x, u \in G,$$

which is precisely the left inverse property. With $u = x^{-1}$ in (6) and by (5), we see that ${}^{-1}x = x^{-1}$ and $(x^{-1})^{-1} = x$. Now (6) and (1) with $x = z^{-1}$ imply

$$(7) \qquad yz^{-1} \cdot z = y, \quad \text{for all} \quad y, z \in G,$$

which is precisely the right inverse property. Thus $G(\cdot)$ is an Inverse property loop.

In (1) replacing y by yx^{-1} and making use of (7), we have

$$yz \cdot \lambda x = yx^{-1} \cdot (x \cdot (z \cdot \lambda x)),$$

that is,

$$A(x) = (R(x^{-1}), L(x)R(\lambda x), R(\lambda x))$$

is an autotopism of $G(\cdot)$. Then — since $G(\cdot)$ is an inverse property loop,

$$B(x) = (L(x), R(\lambda x), L(x)R(\lambda x)), B(x)A(x)^{-1} = (L(x)R(x), L(x)^{-1}, L(x))$$

and $C(x) = (L(x), R(x), L(x)R(x))$ are autotopisms of $G(\cdot)$. Hence, $G(\cdot)$ is a Moufang loop.

Now, we will show that (1) implies (2), (3) and (4) in $G(\cdot)$. $y = 1$ in (1) gives,

$$(8) \qquad xz \cdot \lambda x = x \cdot (z \cdot \lambda x), \quad \text{for all} \quad x, z \in G.$$

Using (8), from autotopy $B(x)$, results

$$xy \cdot (z \cdot \lambda x) = x \cdot [yz \cdot \lambda x] = (x \cdot yz) \cdot \lambda x,$$

which is (2). From (2) and (8), (3) follows. From $B(x)$, we deduce that $D(x) = (L(x)R(\lambda x), L(\lambda x)^{-1}, L(x))$ is an autotopism of $G(\cdot)$. That is,

$$\{x \cdot (y \cdot \lambda x)\} \cdot \{(\lambda x)^{-1} \cdot z\} = x \cdot yz,$$

which by (8) yields

$$(xy \cdot \lambda x) \cdot z = x \cdot (y \cdot (\lambda x \cdot z)),$$

which is (4). Thus (2), (3) and (4) hold in $G(\cdot)$.

Let $\mu: G \to G$ be a mapping such that

$$(9) \qquad x \cdot \mu x = \lambda x.$$

G being Moufang, G is diassociative [1, p. 117] and hence $B(x)C(x)^{-1} = (I, R(\mu x), R(\mu x))$ (using (8)) is an autotopism, showing thereby that $\mu x \in N$ (nucleus).

Conversely, let $G(\cdot)$ be Moufang with nucleus N and μ be defined by (9) such that $\mu x \in N$. Then by (i) follows,

$$(yx\cdot z)\cdot\lambda x = (yx\cdot z)\cdot(x\cdot\mu x) = [(yx\cdot z)\cdot x]\cdot\mu x = [y\cdot(x\cdot zx)]\cdot\mu x,$$
$$= y\cdot[(x\cdot zx)\cdot\mu x] = y\cdot[x\cdot(zx\cdot\mu x)] = y\cdot[x\cdot(z\cdot\lambda x)],$$

which is (1). This completes the proof of this theorem.

Cor. 1. If (1) holds in a loop $G(\cdot)$, then (1) holds in every loop isotopic to $G(\cdot)$.

PROOF. Without loss of generality, let $G(0)$ be a principal isotope of $G(\cdot)$ given by $x0y = xf\cdot f^{-1}y$ [1]. Then $G(0)$ is Moufang and further if $\mu x \in N(\cdot)$, then $\mu x \in N(0)$ [1, p. 56 and 57]. Then by theorem 1, (1) holds in $G(0)$. This completes the proof of this lemma.

Cor. 2. Groups are those Moufang loops which satisfy (1) with $\lambda x = x^2$.

The proof follows immediately from Theorem 1.

Regarding the identity (2), which was considered already in [6], we have the following result. As the proof is very similar to that of Theorem 1 and as most of it is given in [6], we only state the theorem.

Theorem 2. *If (2) is true in a loop $G(\cdot)$, then so are (1), (3) and (4). Further $G(\cdot)$ is Moufang and $\mu x \in N$ (nucleus) where μ is given by (9). Conversely, if $G(\cdot)$ is Moufang and $\mu x \in N$, for all $x \in G$, then the identity (2) and hence (1), (3) and (4) all hold in G.*

Regarding the identity (3), we note the following. Putting $z=1$ in (3) we get (8), from which (2).

Now, we consider the identity (4). But here we require λ to be an onto mapping.

Putting $y = x^{-1}$ in (4), we obtain

$$\lambda x\cdot z = x\cdot(x^{-1}(\lambda x\cdot z)),$$

that is,

$$(10)\qquad u = x\cdot(x^{-1}u), \quad \text{for all} \quad x, u \in G.$$

Setting $u = x$ in (10), we see that ${}^{-1}x = x^{-1}$ and $(x^{-1})^{-1} = x$. So, (10) gives the left inverse property.

$z = (\lambda x)^{-1}$ in (4) gives,

$$(xy\cdot\lambda x)\cdot(\lambda x)^{-1} = xy,$$

that is,

$$(11)\qquad (u\cdot\lambda x)\cdot(\lambda x)^{-1} = u, \quad \text{for all} \quad u, x \in G.$$

Since λ is onto, (11) yields the right inverse property. Hence $G(\cdot)$ is an inverse property loop.

Replacing z by $(\lambda x)^{-1}\cdot z$ in (4), to get

$$(xy\cdot\lambda x)\cdot[(\lambda x)^{-1}\cdot z] = x\cdot yz$$

that is, $A_1(x) = (R(\lambda x)L(x), L(\lambda x)^{-1}, L(x))$ and hence $B_1(x) = (L(x), R(\lambda x), R(\lambda x)L(x))$, are autotopisms of G. Since $B_1(x) = B(x)$, (1) holds in $G(\cdot)$. Then by using Theorem 1, we have proved the following theorem.

Theorem 3. *If λ is an onto mapping of a loop $G(\cdot)$ to itself and if the identity* (4) *is satisfied in $G(\cdot)$, then $G(\cdot)$ is Moufang, in which* (1), (2) *and* (3) *are also satisfied. Further $\mu x \in N$ (nucleus), for all $x \in G$, where μ is given by* (9). *Conversely, if $G(\cdot)$ is Moufang and $\mu x \in N$, for all $x \in G$, then* (4) *holds in G.*

The following corollary is immediate.

Cor. 3. If λ is surjective and (4) holds in a loop $G(\cdot)$, then every loop isotopic to G has property (4).

2. Extra loop type

In a loop $G(\cdot)$, the following identities are equivalent

$$\text{(v)} \qquad yx \cdot zx = (y \cdot xz) \cdot x$$

$$\text{(vi)} \qquad (xy \cdot z) \cdot x = x \cdot (y \cdot zx)$$

$$\text{(vii)} \qquad xy \cdot xz = x \cdot (yx \cdot z).$$

A loop $G(\cdot)$ satisfying any one (and hence all) of these identities is called an extra loop [4].

Let $\alpha: G \to G$ be any mapping. Let us consider the following identities in $G(\cdot)$. Similar to the above identities:

$$(12) \qquad yx \cdot (z \cdot \alpha x) = (y \cdot xz) \cdot \alpha x$$

$$(13) \qquad (xy \cdot z) \cdot \alpha x = x \cdot \big(y \cdot (z \cdot \alpha x)\big)$$

$$(14) \qquad xy \cdot (\alpha x \cdot z) = x \cdot \big((y \cdot \alpha x) \cdot z\big).$$

We prove the following results regarding these identities. First let us take up the identity (12).

Theorem 4. *Let $G(\cdot)$ be a loop and let $\alpha: G \to G$ be any mapping such that the identity* (12) *holds in $G(\cdot)$. Then $G(\cdot)$ is Moufang and* (13) *and* (14) *also hold in $G(\cdot)$. Also $\mu x \in N$ (nucleus) where $\mu: G \to G$ is any mapping such that $\mu x = \alpha x \cdot x$. Conversely, if $G(\cdot)$ is Moufang and $\mu x \in N$, then* (12) *and so* (13) *and* (14) *also are satisfied in $G(\cdot)$.*

PROOF. Set $y = {}^{-1}x$ in (12) to get, using (5),

$$(6) \qquad z = {}^{-1}x \cdot xz, \quad \text{for all} \quad x, z \in G.$$

Hence $G(\cdot)$ possesses the left inverse property and

$${}^{-1}x = x^{-1}, \quad (x^{-1})^{-1} = x.$$

Put $y = (xz)^{-1}$ in (12), to obtain, using (6),

$$[(xz)^{-1} \cdot x] \cdot (z \cdot \alpha x) = \alpha x = z^{-1} \cdot (z \cdot \alpha x),$$

giving,

$$(xz)^{-1} \cdot x = z^{-1},$$

that is,

$$(7) \qquad x = xz \cdot z^{-1},$$

which is the right inverse property. So, G has inverse property.

Replacing z by $x^{-1}z$ in (12) and utilizing (6), we have

$$yx\cdot(x^{-1}z\cdot\alpha x)=yz\cdot\alpha x. \tag{15}$$

Thus $A(x)=(R(x),\ R(\alpha x)L(x^{-1}),\ R(\alpha x))$ and hence

$$B(x)=(L(x^{-1}),\ R(\alpha x),\ R(\alpha x)L(x^{-1})),\quad B(x)^{-1}A(x)=(R(x)L(x),\ L(x^{-1}),\ L(x))$$

and

$$C(x)=(L(x),\ R(x),\ R(x)L(x))$$

are autotopisms of $G(\cdot)$. Hence $G(\cdot)$ is Moufang.

$y=1$ in (15) and (6) give,

$$x^{-1}z\cdot\alpha x=x^{-1}\cdot(z\cdot\alpha x). \tag{16}$$

From (15), (16), (iv) and (iii), result

$$\begin{aligned}x^{-1}\cdot(yz\cdot\alpha x)&=x^{-1}\cdot(yx\cdot\{x^{-1}\cdot(z\cdot\alpha x)\})\\&=[(x^{-1}yx)\cdot x^{-1}]\cdot(z\cdot\alpha x)=(x^{-1}y\cdot xx^{-1})(z\cdot\alpha x),\end{aligned}$$

in which changing y to xy and using (6), we obtain

$$(xy\cdot z)\cdot\alpha x=x\cdot(y\cdot(z\cdot\alpha x)),$$

which is (13). Since $B(x)$ is an autotopism, so is $D(x)=(R(\alpha x)L(x)^{-1},\ L(\alpha x)^{-1},\ L(x^{-1}))$ and so is $D(x)^{-1}$. That is

$$\{x\cdot[y\cdot(\alpha x)^{-1}]\}\cdot(\alpha x\cdot z)=x\cdot yz, \tag{17}$$

is true, from which by replacing y by $y\cdot\alpha x$ and using (7), we obtain

$$xy\cdot(\alpha x\cdot z)=x\cdot((y\cdot\alpha x)\cdot z)$$

which is (14). Let $\mu\colon G\to G$ be a mapping such that

$$\mu x=\alpha x\cdot x. \tag{18}$$

G being Moufang, G is diassociative and therefore $B(x)\cdot C(x)=(I,\ R(\mu x),\ R(\mu x))$ is an autotopism of $G(\cdot)$, that is $\mu x\in N$ (nucleus).

Conversely, let G be Moufang and $\mu x\in N$. In (i) replacing z by zx^{-1}, we get

$$yx\cdot zx^{-1}=(y\cdot xz)\cdot x^{-1},\quad \text{for all}\quad x,y,z\in G. \tag{19}$$

Thus from (19) and the hypothesis that $\mu x\in N$, we have

$$\begin{aligned}yx\cdot(z\cdot\alpha x)&=yx\cdot(z\cdot(\mu x\cdot x^{-1}))=yx\cdot[(z\cdot\mu x)\cdot x^{-1}]=\\&=[y\cdot(x\cdot(z\cdot\mu x))]\cdot x^{-1}=[y\cdot(xz\cdot\mu x)]\cdot x^{-1}=\\&=(y\cdot xz)\cdot(\mu x\cdot x^{-1})=(y\cdot xz)\cdot\alpha x,\end{aligned}$$

which is precisely (12). This completes the proof of this theorem. Regarding the identity (13), which was already treated in [3], we given the following theorem without proof, as the proof is very similar to that of theorem 4 and as most of it is given in [3].

Theorem 5. *Let $G(\cdot)$ be a loop, satisfying* (13). *Then G is Moufang and the identities* (12) *and* (14) *also hold in G. Further $\mu x \in N$ (nucleus), where $\mu x = \lambda x \cdot x$. Conversely, if G is Moufang and $\mu x \in N$, then* (13) *and so* (12) *and* (14) *all hold in G.*

Finally we deal with (14). But here, we take $\alpha: G \to G$ to be surjective.

Letting $z = (\alpha x)^{-1}$ in (14), we have, $y = (y \cdot \alpha x)(\alpha x)^{-1}$, which, since α is onto, implies the right inverse property. As before, we get $x^{-1} = {}^{-1}x$ and $(x^{-1})^{-1} = x$.

Replacing z by $(y \cdot \alpha x)^{-1}$ in (14), we obtain

$$xy \cdot [\alpha x \cdot (y \cdot \alpha x)^{-1}] = x,$$

in which changing x into xy^{-1} and using (7), we have

$$\alpha(xy^{-1}) \cdot [y \cdot \alpha(xy^{-1})]^{-1} = y^{-1}. \tag{20}$$

Change x to xy in (20), to get using (7),

$$\alpha x = y^{-1} \cdot (y \cdot \alpha x). \tag{21}$$

Again using α onto, from (21) follows the left inverse property. Therefore, G has inverse property.

Now (14) can be rewritten as (17) by changing y into $y \cdot (\alpha x)^{-1}$. Thus $B(x)^{-1}$ ($B(x)$ as in Theorem 5) is an autotopism of G, showing thereby that (12) holds in G. Then with the use of Theorem 4, we have the following theorem.

Theorem 6. *If the identity* (14) *holds in a loop $G(\cdot)$ where the mapping $\alpha: G \to G$ is surjective, then* (12) *holds. Further $G(\cdot)$ is Moufang,* (13) *holds in G and $\mu x \in N$* (nucleus) *where $\mu x = \alpha x \cdot x$. Conversely, if $G(\cdot)$ is Moufang and $\mu x \in N$, then* (12), (13) *and* (14) *all hold in G.*

3. Characterizations

In this section we give characterizations for Moufang loops, extra loops and groups with the use of results in sections 1 and 2. First we prove a theorem inter-linking the theorems in sections 1 and 2, which will be the basis for characterizations.

Theorem 7. *Let $G(\cdot)$ be a loop and $\lambda, \mu, \alpha: G \to G$ satisfying* (9) *and* (18), *that is, $\lambda x = x \cdot \mu x = x \cdot (\alpha x \cdot x)$. Then the following conditions are equivalent:*

(M) *Identity* (1) *or* (2) *or* (3) *is satisfied for all $x, y, z \in G$.*

(ME) *$G(\cdot)$ is Moufang and $\mu x \in N$ (nucleus) for all $x \in G$.*

(E) *Identity* (12) *or* (13) *holds for all $x, y, z \in G$.*

Proof. (M) is equivalent to (ME) follows from Theorems 1 and 2. The equivalence of (ME) and (E) results from Theorems 4 and 5. That is, we have $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (12) \Leftrightarrow (13) \Leftrightarrow G$ is Moufang and $\mu x \in N$.

Remark 1. If λ and α are onto mappings, then all in Theorem 7 are equivalent to the identities (4) and (14) individually.

Cor 4. If (1) or (2) or (3) or (12) or (13) holds in a loop, then in every loop isotopic to it also, all hold.

The proof is immediate from Cor. 1 and Theorem 7.

Cor 5. Characterization of Moufang loops. Let $G(\cdot)$ be a loop. Then,

$$G(\cdot) \text{ is Moufang} \Leftrightarrow yx\cdot zx^{-1} = (y\cdot xz)x^{-1} \quad \text{hold in } G(\cdot) \Leftrightarrow$$

$$(\delta) \qquad (xy\cdot z)x^{-1} = x\cdot\big(y\cdot(z\cdot x^{-1})\big) \text{ hold in } G(\cdot)$$

$$\Leftrightarrow xy\cdot(x^{-1}z) = x\cdot\big((y\cdot x^{-1})z\big) \text{ hold in } G(\cdot).$$

PROOF. In Theorem 7, choosing $\lambda x = x$ and $\alpha x = x^{-1}$, and from Remark 1, we conclude that the above equivalences are indeed true. Note that the equivalence with (δ) has been proved in [3].

Cor 6. Extra loops. Let $G(\cdot)$ be a loop. Then $G(\cdot)$ is an extra loop $\Leftrightarrow G(\cdot)$ is Moufang and $x^2 \in N \Leftrightarrow$. Identity (1) or (2) or (3) holds with $\lambda x = x^3$.

PROOF. From Theorem 7, by choosing $\alpha x, = x$, $\mu x = x^2$, $\lambda x = x^3$, the above results follow immediately. Note that the equivalence of the first two and (1) in the third had been proved in [5], [3].

Cor 7. Groups. Let $G(\cdot)$ be a loop. Then $G(\cdot)$ is a group $\Leftrightarrow$. Identity (1) or (2) or (3) holds with $\lambda x = x^2$.

PROOF. This follows as an immediate consequence of Theorem 7 by choosing $\lambda x = x^2$, $\mu x = x$ and $\lambda x = 1$. Thus groups are precisely those Moufang loops which satisfy the M_2- law [6], [2].

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(Received January 5, 1974.)