On minimal pure subgroups

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In [4], L. FUCHS poses the problem of characterizing the subgroups of an abelian group G, which are intersections of finitely many pure subgroups of G (problem 13 p. 134). In considering this problem, we are naturally brought to think of those subgroups which are contained in no proper pure subgroups of G. We establish, in the first section of this paper, a necessary and sufficient condition for a pure subgroup K of G to be a minimal pure subgroup containing a subgroup N. In the second section, we make some applications of this characterization to obtain new and shorter proofs of some results contained in [6] and in [8], as well as some new results connecting these ideas with direct sums of cyclic groups. The notation is that of [4] all groups are abelian, Z^+ denotes the set of non-negative integers, and p denotes a fixed positive prime number. The symbol $\oplus_c$ denotes direct sums of cyclic groups.

1. Minimal pure subgroups

Definition 1.1. A subgroup N of a group G is said to be almost-dense in G (abbreviated a.d.) if, for every pure subgroup K of G containing N, G/K is divisible.

Notation 1.2. Let K be any primary group, for every $n \in Z^+$ we let

$$K_n = (p^n K)[p] = \{x \in p^n K \mid px = 0\}.$$

Theorem 1.3. *Let N be a subgroup of a primary group G. There is no proper pure subgroup of G containing N if and only if N is almost-dense in G, and $N \supset G_n$ for some $n \in Z^+$.*

PROOF. Suppose N is a.d. and $N \supset G_n$ for some $n \in Z^+$. Then, every pure subgroup K of G, containing N, contains also $p^n G$ (see [1], Corollary 3, p. 342). Thus, G/K is at the same time divisible and bounded. And therefore $G/K=0$, i.e. $G=K$. Conversely, if no proper pure subgroup of G contains N, clearly N is a.d. in G. Also, by Lemma 4.1 and theorem 3.7 in [2], $N \supset G_n$ for some $n \in Z^+$.

In view of the preceeding theorem, we need only characterize almost dense subgroups of G. We need the following two lemmas.

Lemma 1.4. *Let G be a primary group and S a subgroup of $G[p]$ such that $S\cap p^nG=0$, for some $n\in Z^+$. Then there exists a pure subgroup K of G such that $K[p]=S$. Furthermore*

$$(K\oplus p^nG)/p^nG \text{ is a pure subgroup of } G/p^nG.$$

PROOF. S can be extended to a p^nG-high subgroup H of G. H is then, pure and bounded (see [4] Theorem 27.7) and there exists a pure subgroup K of H such that $K[p]=S$ (see [7] ex. 18, a). It is easy to check that $K\oplus p^nG/p^nG$ is pure in G/p^nG.

Lemma 1.5. *Let N be a subgroup of a primary group G, such that for some $n\geqq 1$, $N+p^nG\not\supseteq G_{n-1}$. Then, there exists a proper pure subgroup R of G such that:*

$$R\supseteq N+p^nG, \quad \text{and,} \quad N+p^nR\supseteq R_{n-1}.$$

PROOF. Let $G_{n-1}=T\oplus S$, where $T=(N+p^nG)\cap G_{n-1}$. Since $S\cap p^nG=0$ there exists, by lemma 1.4, a pure subgroup K of G, such that : $K[p]=S$, and $(K\oplus p^nG)/p^nG$ is a pure subgroup of G/p^nG, whose socle is $(S\oplus p^nG)/p^nG$. Now, we have:

$$0=p^n(G/p^nG)\subseteq (S\oplus p^nG)/p^nG\subseteq p^{n-1}(G/p^nG).$$

Thus, $(K\oplus p^nG)/p^nG$ is an absolute summand of G/p^nG (see [1] p. 344). Let R/p^nG be a complementary summand of $(K\oplus p^nG)/p^nG$, in G/p^nG, containing $(N+p^nG)/p^nG$. Then $G=R\oplus K$, where $K\neq 0$. Now, since $R\supseteq p^nG$, we have $p^nG=R\cap p^nG=p^nR$. Also $G_{n-1}\supseteq R_{n-1}=R\cap G_{n-1}\supseteq T$, therefore $R_{n-1}=T\oplus(S\cap R_{n-1})$, but $S\cap R_{n-1}=0$ so $R_{n-1}=T$, and we have

$$N+p^nR=N+p^nG\supseteq T=R_{n-1}, \text{ as stated.}$$

Now we use these lemmas to obtain a characterization of almost dense subgroups of a primary group G.

Theorem 1.6. *A subgroup N of a primary group G is almost-dense in G if and only if*

$$(*) \qquad N+p^nG\supseteq G_{n-1} \quad \text{for all} \quad n\geqq 1.$$

PROOF. Suppose N satisfies $(*)$ and K is a pure subgroup of G containing N. If G/K is not divisible it must have a cyclic summand $R/K\neq 0$ (see [7] theorem 9). Now, $G/K=H/K\oplus R/K$ and $G/H\simeq R/K$ is finite so that for some $n\in Z^+$, $p^nG\subseteq H$. Thus, $H\supseteq N+p^nG\supseteq G_{n-1}$ and H being pure we have $H\subseteq p^{n-1}G$. After a finite number of steps we see that $H\supseteq N+pG\supseteq G[p]$, i.e. $H=G$, and $R/K=0$. This is a contradiction, therefore G/K is divisible and N is a.d. in G. Conversely, if $(*)$ is not satisfied then we are in the situation of lemma 1. 5 and N cannot be a. d. since there exists a proper pure subgroup $R\supseteq N+p^nG$ and $G/R\neq 0$ is bounded. Therefore if N is a.d. $(*)$ is satisfied.

Combining theorem 1.3 and theorem 1.6 we obtain.

Theorem 1.7. *Let K be a pure subgroup of a primary group G, containing a subgroup N of G. K is a minimal pure subgroup of G containing N if and only if*

$$N\supseteq K_n \quad \text{for some} \quad n\in Z^+ \quad \text{and} \quad N+r^nK\supseteq K_{r-1} \quad \text{for} \quad 1\leqq r.$$

2. Some applications

We apply now the results in the first section to obtain the following useful criterion.

Theorem 2.1. *Let N be a subgroup of a primary group G. N is contained in a minimal pure subgroup of G if and only if there exists a pure subgroup K of G such that*

$$K \supseteqq N \supseteqq K_n \quad \textit{for some} \quad n \in Z^+.$$

PROOF. If N is contained in a minimal pure subgroup of G, the result follows from theorem 1.7. Conversely, suppose that there exists a pure subgroup K of G such that $K \supseteqq N \supseteqq K_n$ for some $n \in Z^+$. If $n=0$, K itself is minimal pure containing N. If $n \geqq 1$, for every pure subgroup R of K containing N, define.

$$E(R) = \{r \geqq 1 | N + p^r R \not\supseteqq R_{r-1}\}$$

and set $m(R) = 0$ if $E(R) = \emptyset$

$$m(R) = \max \{x \in E(R)\} \quad \text{if} \quad E(R) \neq \emptyset$$

note that, $m(R) \leqq n$ and therefore there exists a pure subgroup H of G containing N for which $m(H)$ is minimal. By lemma 1.5 we see that $m(H)=0$, i.e.

$$H \supseteqq N \supseteqq H_n \quad \text{and} \quad N + p^r H \supseteqq H_{r-1} \quad \text{for all} \quad r \geqq 1.$$

Thus by theorem 1.7, H is a minimal pure subgroup containing N. This completes the proof.

Remark. The preceeding criterion was first, to our knowledge, formulated by J. D. MOORE in his doctoral dissertation ([8] p. 45). His proof however, was based on an unproved conjecture ([8] p. 41) whose validity has not been established anywhere that we know of.

Corollary 2.2. ([6]) In a bounded primary group every subgroup is contained in a minimal pure subgroup.

To conclude this section, we look at the important special cases where $G/N = \oplus_c$, or where N is contained in a basic subgroup of G. In [3], we proved that if a subsocle S of a primary group G supports a pure subgroup K of G, then $G/S = \oplus_c$ implies $G/K = \oplus_c$. Our next result generalizes this fact.

Theorem 2.3. *Let N be a subgroup of a primary group G, such that $G/N = \oplus_c$. If K is a minimal pure subgroup of G containing N then $G/K = \oplus_c$.*

PROOF. We know that $N \supseteqq K_n$ for some $n \in Z^+$. Now, since K is pure in G, $p^n(G/K)[p] = (G_n + K)/K$, and the natural homomorphism from G/N onto G/K maps $(G_n + N)/N$ onto $(G_n + K)/K$. This map is height preserving, indeed for $g \in G_n$

$$h(g+K) \geqq n \quad \text{and if} \quad g + K = p^m x + K \quad \text{where} \quad m \geqq n,$$

the purity of K allows us to choose $p^m x \in G[p]$, thus $g - p^m x \in K[p]$. Now, $h_G(g - p^m x) \geqq n$ and $h_K(g - p^m x) \geqq n$, therefore $g - p^m x \in K_n \subset N$, so that $g + N = p^m x + N$ and $h(g+K) \leqq h(g+N)$. Since $(G_n + N)/N$ is the union of an ascending chain of subgroups of bounded height in G/N, we conclude that $(G_n + K)/K$ has the same property in G/K. Thus, $p^n(G/K) = \oplus_c$ (see [7] theorem 12) and $G/K = \oplus_c$ (see [4] prop. 18, 3 p. 92). This completes the proof.

A close examination of the preceeding proof yields the following.

Corollary 2.4. Let N be a subgroup of a primary group G, such that $G/N = \oplus_c$ and let K be a pure subgroup of G, such that: $K \supseteqq N \supseteqq K_n$, for some $n \in Z^+$, then K is a summand of G.

The final result in this section is in some sense dual to theorem 2.4.

Theorem 2.5. *Let N be a subgroup of a basic subgroup B of a primary group G. If N is contained in a minimal pure subgroup K of G then $K = \oplus_c$.*

Proof. Since $N \subseteqq B$, N satisfies the conditions of theorem 29.5 p. 99 in [5], and since K is pure, N can be expanded to a basic subgroup H of K. Since K is minimal pure containing N, $H = K$ and therefore $K = \oplus_c$ as stated.

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