Extensions of almost diffuse, countably almost diffuse and c_0-operators

By E. TARAFDAR (St. Lucia, Qld)

In [1] FOIAIS and SINGER have introduced the classes of almost diffuse and countably almost diffuse operators mapping a space $C([0, 1])$ into a Banach space X.

WHITLEY [2] has shown the validity of the results of [1] when S is an arbitrary compact Hausdorff space and has also introduced a class of operators called c_0-operators mapping a space $C(S)$ into a Banach space. All these classes of operators include compact and weakly compact operators and have many other properties (see [1] and [2]). The following wellknown result is due to LINDENSTRAUSS [3]:

The second conjugate X^{**} of a Banach space is a P_λ space $\Leftrightarrow$ each weakly compact (resp. compact) operator from X into a Banach space Y has a weakly compact (resp. compact) extension $\tilde{T}$ from $Z(Z \supset X)$ into Y with $\|\tilde{T}\| \leqq \lambda \|T\|$. (For definition of a P_λ space, $\lambda \geqq 1$ see DAY [4], p. 94.)

Since almost diffuse, countably almost diffuse and c_0-operators are defined only on the spaces $C(S)$ and it is wellknown that $C(S)^{**}$ is a P_1 space for any compact Hausdorff space S (e.g. see GOODNER [5], p. 106), an analogue of the above result of Lindenstrauss for these operators will be the following: Each almost diffuse (resp. countably almost diffuse, resp. c_0-) operator T mapping a space $C(S)$ into a Banach space X has an almost diffuse (resp. a countably almost diffuse, resp. a c_0-) extension $\tilde{T}$ mapping $C(Q)$ $(C(Q) \supset C(S))$ into X with $\|\tilde{T}\| = \|T\|$. In this note we prove this result in the particular case when $Q = \Lambda$, the unit ball of $C(S)^*$.

We now state the the following definitions as given in [1] and [2].

S is a compact Hausdorff space and X any Banach space. The oscillation of a bounded linear operator $T: C(S) \to X$ at a point $s \in S$ is denoted by $W_T(s) = W(T, s)$ and is defined to be the supremum over all positive α satisfying the following: for every neighborhood $\cup$ of s there is a function f, of norm $\leqq 1$ in $C(S)$, which vanishes outside of $\cup$ with $\|Tf\| \geqq \alpha$.

Let $T: C(S) \to X$ be a bounded linear operator. Then T is almost diffuse if the set of diffusion points of T, $D(T) = W_T^{-1}(0)$, is dense. T is countably almost diffuse if the set of concentration points, $\gamma(T) = \{s: s \in S, W_T(s) > 0\}$, is countable and T is c_0-operator if for each $\varepsilon > 0$, the set $\{s: s \in S, W_T(s) > \varepsilon\}$ is finite.

Let Y be a Banach space and Λ be the unit ball in Y^*. Then Λ is a compact Hausdorff space in the weak*-topology (the Y-topology of Y^* in the terminology of DUNFORD and SCHWARZ [6]). Y is isometrically isomorphic to a closed subspace of the Banach space $C(\Lambda)$ where $\varkappa$ is the corresponding isometric isomorphism of Y onto a closed subspace of $C(\Lambda)$ and is defined as follows: for each $y\in Y$, $\varkappa(y)=$ the restriction of $\varkappa_1(y)$ to Λ where $\varkappa_1$ is the natural embedding of Y into Y^{**} (see DUNFORD and SCHWARZ [6], con. 3, p. 424). We first prove the following lemma:

Lemma. *For every bound linear operator $T:C(S)\to X$ mapping a space $C(S)$ into an arbitrary Banach space X, there exists a bounded linear operator $\tilde{T}:C(\Lambda)\to X$ such that $\tilde{T}\varkappa=T$ and $\|\tilde{T}\|=\|T\|$ where Λ is the unit ball in $C(S)^*$ and $\varkappa$ has the corresponding meaning as explained above. In other words every bounded linear operator $T:C(S)\to X$ has a bounded linear extension $\tilde{T}:C(\Lambda)\to X$ with $\|\tilde{T}\|=\|T\|$.*

PROOF. By lemma 7 of DUNFORD and SCHWARTZ ([6], p. 442) there is a homeomorphism λ of S onto a subset of Λ where Λ has weak*- topology and λ is defined by $\lambda:s\to x_s^*$ for each $s\in S$ and x_s^* is defined by $x_s^*(f)=f(s)$, $f\in C(S)$. We now construct our required operator $\tilde{T}$.

Let f_1 be an arbitrary element of $C(\Lambda)$. Then the restriction $\tilde{f}_1$ of f_1 to $\lambda(S)$ is a continuous map on $\lambda(S)$ where $\lambda(S)$ has the relative weak*-topology. Thus the composition $\tilde{f}_1\lambda$ is a continuous map on S. We set $\tilde{T}f_1=T\tilde{f}_1\lambda$ for each $f_1\in C(\Lambda)$. Clearly $\tilde{T}$ is well defined and linear. We next prove the boundedness of $\tilde{T}$. We observe that $\|\tilde{f}_1\lambda\|=\sup_{s\in S}|\tilde{f}_1\lambda(s)|=\sup_{x^*\in\lambda(S)}|\tilde{f}_1(x^*)|\leqq\sup_{x^*\in\Lambda}|f_1(x^*)|=\|f_1\|$. Hence $\|\tilde{T}f_1\|=$ $=\|T\tilde{f}_1\lambda\|\leqq\|T\|\,\|\tilde{f}_1\lambda\|\leqq\|T\|\,\|f_1\|$. Hence $\tilde{T}$ is bounded and moreover

$$\|\tilde{T}\|\leqq\|T\|. \tag{1}$$

Next, we prove that $\tilde{T}\varkappa=T$. Let $f\in C(S)$ be arbitrary and $\varkappa f=g$. We first see that $\tilde{g}\lambda=f$ where $\tilde{g}$ is the restriction of g to $\lambda(S)$. For each $s\in S$,

$$\begin{aligned}\tilde{g}\lambda(s)&=\tilde{g}(x_s^*)=g(x_s^*)\ \text{ as }\ x_s^*\in\lambda(S),\\ &=\varkappa f(x_s^*)=x_s^*(f)\ \text{ by definition of }\varkappa\\ &=f(s)\ \text{ by definition of }\ x_s^*.\end{aligned}$$

Thus $\tilde{g}\lambda=f$. Hence $\tilde{T}\varkappa f=\tilde{T}g=T(\tilde{g}\lambda)=Tf$. Now $\|T\|=\|\tilde{T}\varkappa\|\leqq\|\tilde{T}\|\,\|\varkappa\|=\|\tilde{T}\|$. Combining this with (1), we obtain $\|\tilde{T}\|=\|T\|$. This completes the proof.

Let $T:C(S)\to X$ be any bounded linear operator. We suppose that $\tilde{T}$ hasbeen obtained as in the above lemma. Remembering all the notations used in the proof of the above lemma we note the following facts:

(1) Every point $y\in[\lambda(S)]^c$ (complement of $\lambda(S)$ in Λ) is a diffusion point of $\tilde{T}$, i.e., $W_{\tilde{T}}(y)=0$ for each $y\in[\lambda(S)]^c$.

PROOF. Let $\varepsilon>0$ be any number. Now $0=[\lambda(S)]^c$ is a neighborhood of y in Λ. Let f be any function in $C(\Lambda)$ such that $\|f\|\leqq1$ and f vanishes outside of 0. Obviously $\tilde{f}\lambda(s)=0$ for each $s\in S$, where $\tilde{f}$ is the restriction of f to $\lambda(S)$ as before. Hence $\|\tilde{T}f\|=\|T\tilde{f}\lambda\|=0<\varepsilon$. Thus $(W_{\tilde{T}}y)=0$. (Notice that same 0 works for each $\varepsilon>0$).

(2) If y is a concentration point of $\tilde{T}$, i.e., $y\in\gamma(\tilde{T})$, then (i) $y\in\gamma(\tilde{T})\cap\lambda(S)$, (ii) $s=\lambda^{-1}(y)\in\gamma(T)$ and (iii) furthermore if $W_{\tilde{T}}(y)>\alpha$, $\alpha>0$, then $W_T(s)>\alpha$.

PROOF. (i) follows from (1). Let $W_{\tilde{T}}(y)>\varepsilon$, $\varepsilon>0$. Let N be a neighborhood of s in S. Then $\lambda(N)$ is a neighborhood of y in $\lambda(S)$. We can find a neighborhood Q of y in Λ such that $Q\cap\lambda(S)=\lambda(N)$. Since $W_{\tilde{T}}(y)>\varepsilon$. There exists a function f in $C(\Lambda)$ such that f vanishes outside of Q, $\|f\|\leqq1$ and $\|\tilde{T}f\|>\varepsilon$. Let $\tilde{f}$ be the restriction of f to $\lambda(S)$. Obviously $\tilde{f}$ vanishes outside of $\lambda(N)$ in $\lambda(S)$ (i.e. outside of $\lambda(N)$ relative to $\lambda(S)$). We consider the function $\tilde{f}\lambda$ in $C(S)$. Clearly $\tilde{f}\lambda$ vanishes outside of N and moreover as in the proof of the lemma, $\|\tilde{f}\lambda\| \leqq \|f\| \leqq 1$. Also as in the proof the lemma $\|T\tilde{f}\lambda\| = \|Tf\| > \varepsilon$. Thus it follows that $W_T(s) > \varepsilon$ and $s \in \gamma(T)$. Hence both (ii) and (iii) are proved.

(3) If s is a diffusion point of T, *i.e.*, if $s \in D(T)$ then $\lambda(s) \in D(\tilde{T})$.

PROOF. That $\lambda(s) \in D(T)$ follows directly from (2). We are now in a position to prove the following theorem:

Theorem. *For each almost diffuse (resp. countably almost diffuse, resp. c_0-) operator $T: C(S) \to X$ mapping a space $C(S)$ into an arbitrary Banach space X, there is an almost diffuse (resp. a countably almost diffuse, resp. a c_0-) operator $\tilde{T}: C(\Lambda) \to X$ such that $\tilde{T}\varkappa = T$ and $\|\tilde{T}\| = \|T\|$, where Λ and $\varkappa$ have the same meaning as in the above lemma.*

PROOF. By the above lemma, for each bounded linear operator $T: C(S) \to X$, there is a bounded linear operator $\tilde{T}: C(\Lambda) \to X$ such that $\tilde{T}\varkappa = T$ and $\|\tilde{T}\| = \|T\|$. We prove that $\tilde{T}$ is the required operator in each case.

Let T be almost diffuse. Then $D(T)$ is dense in S. Hence $\lambda(D(T))$ is densein $\lambda(S)$ as λ is a homeomorphism. Evidently $\lambda(D(T)) \cup [\lambda(S)]^c$ is dense in Λ. By using the facts (1) and (3), we have $\lambda D(T) \cup [\lambda(S)]^c \subseteqq D(\tilde{T})$. Hence $D(\tilde{T})$ is dense in Λ. Thus $\tilde{T}$ is almost diffuse.

Next, let T be countably almost diffuse. Then $\gamma(T)$ is countable. By the fact 2 (ii) it follows that $\lambda^{-1}(\gamma(\tilde{T})) \subseteqq \gamma(T)$. Now since λ is a homeomorphism, it follows that $\gamma(\tilde{T})$ is countable. Hence $\tilde{T}$ is countably almost diffuse.

Finally, let T be c_0-operator. If possible, suppose that $\tilde{T}$ is not a c_0-operator. Then there is a number $\alpha > 0$ such that the set $A = \{y: y \in \Lambda, W_{\tilde{T}}(y) > \alpha\}$ is infinite. By the fact 2 (i), $A = \{y: y \in \lambda(S), W_{\tilde{T}}(y) > \alpha\}$. Now $\lambda^{-1}(A) = \{s: s \in S, \lambda(s) \in A\} = \{s: s \in S, \lambda(s) \in A, W_T(s) > \alpha\}$, by 2 (iii), $\subseteqq \{s: s \in S, W_T(s) > \alpha\}$. Since $\lambda^{-1}(A)$ is infinite, λ being a homeomorphism, it follows that $\{s: s \in S, W_T(s) > \alpha\}$ is infinite. This contradicts that T is a c_0-operator. Hence $\tilde{T}$ is a c_0-operator.

Remark. Note that all the above results will also hold if Λ is replaced by the weak*-closure of the set of extremal points of Λ.

References

[1] C. FOIAS and I. SINGER, Points of diffusion of linear operators and almost diffuse operators in spaces of continuous functions, *Math. Z.* **87** (1965), 434—450. MR31, 5093.

[2] R. WHITLEY, The oscillation of an operator, *Trans. Amer. Math. Soc.* **165** (1972), 65—73.

[3] J. LINDENSTRAUSS, Extension of compact operators, *Mem. Amer. Math. Soc.* **48** (1964). MR31, 3828.

[4] M. DAY, Normed linear spaces, Academic Press, *New York; Berlin*, 1962. MR26, 2847.

[5] D. B. GOODNER, Projection in normed linear spaces, *Trans. Amer. Math. Soc.* **69** (1950), 89—108. MR12, 266.

[6] N. DUNFORD and J. T. SCHWARTZ, Linear operators. I: (General theory, pure and appl. math., Vol. 7), *New York* and *London*, 1958. 22, 8302.

UNIVERSITY OF QUEENSLAND,
ST. LUCIA, QLD.

(*Received 12. February, 1974.*)