Some remarks on a generalization of the Stieltjes transform

By H. M. SRIVASTAVA (Victoria, B. C.)

1. The integral equation

$$\varphi\{f(x):p\} = \int_0^\infty (x+p)^{-1} f(x)\,dx \tag{1}$$

is said to define the Stieltjes transform of a function $f(x)$, provided that the integral on the right-hand side exists. We say that $\varphi\{f(x):p\}$ is the image of $f(x)$ under the kernel $(x+p)^{-1}$. A natural generalization of (1) is given in the literature by means of the integral equation (cf. [8], p. 30)

$$\varphi^\lambda\{f(x):p\} = \int_0^\infty (x+p)^{-\lambda} f(x)\,dx, \quad \lambda>0, \tag{2}$$

and whenever this integral exists, $\varphi^\lambda\{f(x):p\}$ is called the generalized Stieltjes transform of $f(x)$ of order λ.

Recently, J. L. ARORA [1] has introduced a further generalization of (2) given by

$$\varphi_m^\lambda\{f(x):p\} = \int_0^\infty (x^m+p^m)^{-\lambda} f(x)\,dx, \quad \lambda, m, p>0, \tag{3}$$

provided that the second member of (3) exists. Evidently, one has

$$\varphi_1^\lambda\{f(x):p\} = \varphi^\lambda\{f(x):p\} \quad \text{and} \quad \varphi_1^1\{f(x):p\} = \varphi\{f(x):p\}, \quad \lambda, p>0, \tag{4}$$

in the notations of (1), (2) and (3).

It may be of interest to remark that the generalization (3) of the integral transform (2) is rather trivial; indeed it is easily verified that

$$\varphi_m^\lambda\{f(x):p\} = m^{-1}p^{1-m\lambda}\varphi^\lambda\{x^{(1-m)/m}f(px^{1/m}):1\}, \tag{5}$$

where $\lambda, m, p>0$.

The linear relationship (5) between the generalized Stieltjes transform (2) and its subsequent variation (3) studied by Arora [1] can be used to convert readily available tables of $\varphi^\lambda\{f(x):p\}$ into those of $\varphi_m^\lambda\{f(x):p\}$. As an illustration, we notice that if

$$f(x) = H_{u,v}^{l,n}\left[zx\left|\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_u, A_u)\\(\beta_1, B_1), \dots, (\beta_v, B_v)\end{matrix}\right.\right], \tag{6}$$

where $H_{u,v}^{l,n}[z]$ denotes the familiar H-function (cf., e.g., [5], p. 214, Eq. (2.4)), $A_j>0, j=1, \ldots, u, B_j>0, j=1, \ldots, v$, then (cf. [2], p. 237)

$$\varphi^{\lambda}\{f(x):p\}=\frac{p^{1-\lambda}}{\Gamma(\lambda)}H_{u+1,v+1}^{l+1,n+1}\left[zp\left|\begin{matrix}(0,1),(\alpha_1,A_1),\ldots,\alpha_u,A_u)\\(\lambda-1,1),(\beta_1,B_1),\ldots,(\beta_v,B_v)\end{matrix}\right.\right], \tag{7}$$

provided

$$\begin{cases}\lambda>1+\max\limits_{1\le j\le n}\{\mathrm{Re}\,[(\alpha_j-1)/A_j]\}, \quad \min\limits_{1\le j\le l}\{\mathrm{Re}\,(\beta_j/B_j)\}>-1,\\ |\arg(z)|<\dfrac{1}{2}\pi\Delta, \quad \Delta\equiv\sum\limits_{j=1}^{n}A_j-\sum\limits_{j=n+1}^{u}A_j+\sum\limits_{j=1}^{l}B_j-\sum\limits_{j=l+1}^{v}B_j>0.\end{cases} \tag{8}$$

Making use of (5) and (7) we have

$$\varphi_m^{\lambda}\{f(x):p\}=\frac{p^{1-m\lambda}}{m\Gamma(\lambda)}H_{u+1,v+1}^{l+1,n+1}\left[zp\left|\begin{matrix}\left(1-\frac{1}{m},\frac{1}{m}\right),(\alpha_1,A_1),\ldots,(\alpha_u,A_u)\\ \left(\lambda-\frac{1}{m},\frac{1}{m}\right),(\beta_1,B_1),\ldots,(\beta_v,B_v)\end{matrix}\right.\right], \tag{9}$$

valid under conditions in (8) when $f(x)$ is given by (6).

We remark in passing that a large variety of special functions which occur in problems of applied mathematics and mathematical analysis can be expressed in terms of the H-function which reduces, when $A_j=B_k=1, j=1, \ldots, u; k=1, \ldots, v$, to the relatively more familiar G-function and which includes, as a special case, the generalized hypergeometric ${}_u\Psi_v[z]$ function of Wright. Thus the transform pair given by (6) and (9) would apply not only to the simpler special functions of mathematical physics such as the Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, the Bessel functions $J_\nu(z)$ and $I_\nu(z)$, the Whittaker function $M_{\mu,\nu}(z)$, the Wright function $J_\nu^\mu(z)$, the classical orthogonal polynomials of Jacobi, Hermite, Laguerre, etc., which are particular cases of the generalized hypergeometric function ${}_uF_v[z]$ or ${}_u\Psi_v[z]$, but also to the more involved Bessel functions $K_\nu(z)$ and $Y_\nu(z)$, the Whittaker function $W_{\mu,\nu}(z)$, their various combinations and other related functions. Indeed, by making use of extensive tables of functions (or their various combinations) expressible in terms of the G-function or the generalized hypergeometric functions (see, for instance, [2], pp. 434—444; see also [4], pp. 209—234), one can deduce from (6) and (9) a fairly large table of these functions and their respective images under the „generalized" Stieltjes transform (3).

2. Since

$$(x+p)^{-\lambda}=p^{-\lambda}{}_1F_0[\lambda;-;-x/p], \tag{10}$$

it would seem natural to consider further generalizations of the generalized Stieltjes transform (2) by replacing the hypergeometric ${}_1F_0$ function in the kernel by the Gauss function

$${}_2F_1[\lambda,\mu;\nu;-x/p], \tag{11}$$

or more generally, by the hypergeometric function

$${}_uF_v[\alpha_1,\ldots,\alpha_u;\beta_1,\ldots,\beta_v;-x/p], \tag{12}$$

or indeed by the H-function

$$H_{u,v}^{l,n}\left[\frac{x}{p}\left|\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_u, A_u)\\(\beta_1, B_1), \dots, (\beta_v, B_v)\end{matrix}\right.\right]. \tag{13}$$

Admittedly, the literature is full of generalizations in these directions, and we content ourselves by referring the interested readers to the works of R. S. VARMA [7], J. M. C. JOSHI [3], and R. SWAROOP [6], where kernels of type (11) have been used to generalize (2). As a special case of the results of these papers one can easily deduce a complex inversion formula for the generalized Stieltjes transform (2); thus, under certain conditions, we have (cf. [8], p. 31, Theorem 8.1)

$$f(x) = \frac{\Gamma(\lambda)}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\frac{\Phi^\lambda(s)x^{\lambda-s-1}}{\Gamma(s)\Gamma(\lambda-s)}\,ds, \quad \tau > 0, \tag{14}$$

where, for convenience, $\Phi^\lambda(s)$ denotes the Mellin transform of $\varphi^\lambda\{f(x):p\}$ given by (2).

More specifically, we have

$$\Phi^\lambda(s) = \int_0^\infty p^{s-1}\varphi^\lambda\{f(x):p\}\,dp, \tag{15}$$

and on substituting from (2) it is seen fairly simply that

$$\Phi^\lambda(s) = \frac{\Gamma(s)\Gamma(\lambda-s)}{\Gamma(\lambda)}\int_0^\infty x^{s-\lambda}f(x)\,dx, \tag{16}$$

provided this last integral is convergent. Similarly, if $\Phi_m^\lambda(s)$ denotes the Mellin transform of $\varphi_m^\lambda\{f(x):p\}$, we have

$$\Phi_m^\lambda(s) = \frac{\Gamma(s/m)\Gamma(\lambda-s/m)}{m\Gamma(\lambda)}\int_0^\infty x^{s-m\lambda}f(x)\,dx. \tag{17}$$

Now an elementary change of the variable of integration in (16) would yield

$$\Phi^\lambda(s/m) = \frac{m\Gamma(s/m)\Gamma(\lambda-s/m)}{\Gamma(\lambda)}\int_0^\infty x^{s-m\lambda}\{x^{m-1}f(x^m)\}\,dx, \tag{18}$$

which, in conjunction with (17), exhibits the obviously well-anticipated fact that the inversion formula (14) can be rewritten at once in terms of the "generalized" Stieltjes transform (3) if we replace s by s/m, $x^{m-1}f(x^m)$ by $f(x)$, and $\Phi^\lambda(s/m)$ by $m^2\Phi_m^\lambda(s)$. Thus, as a trivial variation of (14), we obtain a complex inversion formula for (3) given by

$$f(x) = \frac{m\Gamma(\lambda)}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\frac{\Phi_m^\lambda(s)x^{m\lambda-s-1}}{\Gamma(s/m)\Gamma(\lambda-s/m)}\,ds, \quad \tau > 0, \tag{19}$$

where, as also in (14), $f(x)$ must be replaced by $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at a point $t=x$ such that

$$\lim_{t\downarrow x} f(t) = f(x+0) \neq f(x-0) = \lim_{t\uparrow x} f(t). \tag{20}$$

This formula (19) is substantially what is contained in Theorem 2, p. 108 of ARORA [1] who does also state (14) as a „special'' case of (19) when $m=1$. One can similarly verify that almost every property of the "generalized" Stieltjes transform (3), given by Arora [op. cit.], would follow rather trivially from the corresponding property of the generalized Stieltjes transform (3) or its nontrivial generalizations studied systematically in the literature (cf., e.g., [3], [6] and [7]).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA
CANADA V8W 2Y2

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