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Nonoscillation in half-linear differential equations

By HORNG-JAAN LI (Taiwan) and CHEH-CHIH YEH (Taiwan)

Abstract. We establish some necessary conditions on the nonoscillation of the following half-linear second order differential equation

$$[r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \qquad t \geq t_0,$$

where $p > 1$ is a constant, $r(t)$ and $c(t)$ are continuous functions from $[t_0, \infty)$ to $[0, \infty)$ with $r(t) > 0$.

1. Introduction

This paper is concerned with the half-linear second order differential equation

$$\text{(E)} \qquad [r(t)|u'(t)|^{p-2}u'(t)]' + c(t)|u(t)|^{p-2}u(t) = 0, \qquad t \geq t_0,$$

where $p > 1$ is a constant, $r(t)$ and $c(t)$ are continuous functions on $[t_0, \infty)$ for some $_0 \geq 0$. Throughout the paper, we assume that

(A_1) $\frac{1}{p} + \frac{1}{q} = 1$;
(A_2) $r(t) > 0$ for $t \geq t_0$ and $\int_{t_0}^{\infty} r^{1-q}(s)ds = \infty$;
(A_3) $c(t) \geq 0$ for $t \geq t_0$ and $c(t) \not\equiv 0$ on any interval of the form $[t, \infty)$, $t \geq t_0$.

By a solution of (E) we mean a function $u \in C^1[t_0, \infty)$ such that $r|u'|^{p-2}u' \in C^1[t_0, \infty)$ and that satisfies (E). In [1], Elbert established the existence, uniqueness and extension to $[t_0, \infty)$ of solutions to the initial value problem for (E). We will say that a nontrivial solution u of (E) is

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nonoscillatory if there exists a number $N > 0$ such that $u(t) \neq 0$ for all $t \geq N$. Equation (E) is nonoscillatory if all its solutions are nonoscillatory.

Kusano, Naito and Ogata [2], and Li and Yeh [3] independently showed that if (E) is nonoscillatory then

$$\int_{t_0}^{\infty} c(s)ds < \infty \tag{1}$$

and

$$\limsup_{t\to\infty} \pi^{p-1}(t) \int_t^{\infty} c(s)ds \leq 1, \tag{2}$$

where

$$\pi(t) = \int_{t_0}^{t} r^{1-q}(s)ds, \quad t \geq t_0.$$

It follows from (2) that if (E) is nonoscillatory then

$$\int_{t_0}^{\infty} r^{1-q}(s) \left(\int_s^{\infty} c(\tau)d\tau \right)^q ds < \infty. \tag{3}$$

The purpose of this paper is to improve the results (1), (2), (3), and hence extend the result of Lovelady [4].

2. Main results

In order to prove our main theorem, we need the following lemma.

Lemma 2.1. *If $u(t)$ is a nonoscillatory solution of* (E) *which is not eventually a constant, then $u(t)u'(t) > 0$ for all large t.*

Proof. Without loss of generality, we may assume that $u(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. It follows from (E) that

$$[r(t)|u'(t)|^{p-2}u'(t)]' \leq 0 \quad \text{for } t \geq T_0, \tag{1}$$

which implies that $r(t)|u'(t)|^{p-2}u'(t)$ is nonincreasing on $[T_0, \infty)$. Suppose there exists a $T_1 \geq T_0$ such that $u'(T_1) \leq 0$. Then $r(T_1)|u'(T_1)|^{p-2}u'(T_1) \leq 0$. Since $r(t)|u'(t)|^{p-2}u'(t)$ is decreasing and not identically zero on $[T_0, \infty)$, there exists a $T_2 \geq T_1$ such that

$$r(t)|u'(t)|^{p-2}u'(t) \leq r(T_2)|u'(T_2)|^{p-2}u'(T_2) = -k < 0 \quad \text{for } t \geq T_2,$$

which implies

$$u'(t) \le -k^{q-1} r^{1-q}(t) \quad \text{for } t \ge T_2. \tag{2}$$

Integrating (2) from T_2 to t, we obtain by (A_2)

$$u(t) \le u(T_2) - k^{q-1} \int_{T_2}^{t} r^{1-q}(s)ds \to -\infty \quad \text{as } t \to \infty,$$

which contradicts to $u(t) > 0$ on $[T_0, \infty)$. Thus $u'(t) > 0$ on $[T_0, \infty)$. This completes our proof.

Theorem 2.2. *Let*

$$f(t) = \int_{t}^{\infty} c(s)ds, \quad t \in [t_0, \infty).$$

If (E) *is nonoscillatory, then there exist a number* $T_0 \ge t_0$ *and a sequence* $\{w_k\}(t)_{k=0}^{\infty}$ *of continuous functions from* $[T_0, \infty)$ *to* $(0, \infty)$ *with the following properties:*

(a) $w_1 = f$.

(b) $w_k(t) \le w_{k+1}(t)$ *for* $t \ge T_0$ *and each integer* $k \ge 1$.

(c) $\int_t^\infty r^{1-q}(s) f^{q-1}(s) w_k(s)ds < \infty$ *for* $t \ge T_0$ *and each integer* $k \ge 0$; *and*
$w_{k+1}(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_k(s)ds$ *for* $t \ge T_0$ *and each integer* $k \ge 1$.

(d) If $t \ge T_0$, *then* $w_0(t) = \lim_{k\to\infty} w_k(t)$, *and the convergence is uniform in each compact subset of* $[T_0, \infty)$.

(e) $\limsup_{t\to\infty} \pi^{p-1}(t) w_k(t) \le 1$ *for each integer* $k \ge 0$.

(f) $w_0(t) = f(t) + (p-1) \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_0(s)ds$ *for* $t \ge T_0$.

Proof. Let $u(t)$ be a nonoscillatory solution of (E). By Lemma 2.1, without loss of generality, we may assume that $u(t) > 0$ and $u'(t) > 0$ on $[T_0, \infty)$ for some $T_0 \ge t_0$. Let

$$w(t) = \frac{r(t)|u'(t)|^{p-2}u'(t)}{|u(t)|^{p-2}u(t)} \quad \text{for } t \ge T_0.$$

Then $w(t) > 0$ and

$$w'(t) = -c(t) - (p-1) r^{1-q}(t) w^q(t) < 0 \tag{3}$$

for $t \geq T_0$. This implies that $w(t)$ is decreasing and $\lim_{t\to\infty} w(t)$ exists. Integrating (3) from t to T, we obtain

$$w(T) - w(t) = -\int_t^T c(s)ds - (p-1)\int_t^T r^{1-q}(s)w^q(s)ds$$

for $T \geq t \geq T_0$. It follows from (1) and the existence of $\lim_{t\to\infty} w(t)$ that

$$\int_{T_0}^{\infty} r^{1-q}(s)w^q(s)ds < \infty. \tag{4}$$

It follows from (A_2) and the decrease of $w(t)$ that $\lim_{T\to\infty} w(T) = 0$. This implies

$$w(t) = f(t) + (p-1)\int_t^{\infty} r^{1-q}(s)w^q(s)ds \quad \text{for } t \geq T_0. \tag{5}$$

It is clear from (5) that $w \geq f$ on $[T_0, \infty)$, and hence (4) and (5) imply that

$$\int_{T_0}^{\infty} r^{1-q}f^{q-1}(s)w(s)ds \leq \int_{T_0}^{\infty} r^{1-q}(s)w^q(s)ds < \infty \tag{6}$$

and

$$w(t) \geq f(t) + (p-1)\int_t^{\infty} r^{1-q}f^{q-1}(s)w(s)ds \tag{7}$$

for $t \geq T_0$, respectively. It follows from (E) that $r^{q-1}(t)u'(t)$ is decreasing on $[T_0, \infty)$. Then

$$\begin{aligned}\frac{u(t)}{r^{1-q}(t)u'(t)\pi(t)} &= \frac{u(T_0) + \int_{T_0}^t u'(s)ds}{r^{q-1}(t)u'(t)\pi(t)} \\ &= \frac{u(T_0) + \int_{T_0}^t r^{1-q}(s)r^{q-1}(s)u'(s)ds}{r^{q-1}(t)u'(t)\pi(t)} \\ &\geq \frac{u(T_0) + r^{q-1}(t)u'(t)\int_{T_0}^t r^{1-q}(s)ds}{r^{q-1}(t)u'(t)\pi(t)} \\ &\geq \frac{\pi(t) - \pi(T_0)}{\pi(t)}\end{aligned}$$

for $t \geq T_0$. This implies that

$$\pi^{p-1}(t)w(t) \leq \left(\frac{\pi(t)}{\pi(t) - \pi(T_0)}\right)^{p-1},$$

thus,

$$\limsup_{t\to\infty} \pi^{p-1}(t)w(t) \le 1. \tag{8}$$

Let $w_1(t) = f(t)$ on $[T_0, \infty)$, and let

$$w_2(t) = f(t) + (p-1)\int_t^\infty r^{1-q}(s)f^{q-1}(s)w_1(s)ds \quad \text{for } t \ge T_0.$$

Then $w_2(t) \ge w_1(t)$ and

$$w_2(t) \le f(t) + (p-1)\int_t^\infty r^{1-q}(s)f^{q-1}(s)w(s)ds \le w(t)$$

for $t \ge T_0$. It follows from (8) that $\limsup_{t\to\infty} \pi^{p-1}(t)w_2(t) \le 1$. Suppose n is a positive integer and $w_1, w_2, \ldots, w_n$ are defined such that $w_1 \le w_2 \le \cdots \le w_n \le w$ on $[T_0, \infty)$, then

$$\int_{T_0}^\infty r^{1-q}(s)f^{q-1}(s)w_k(s)ds < \infty$$

whenever $1 \le k \le n$, and

$$w_{k+1}(t) = f(t) + (p-1)\int_t^\infty r^{1-q}(s)f^{q-1}(s)w_k(s)ds$$

whenever $1 \le k \le n-1$ and $t \ge T_0$. Let w_{n+1} be given by

$$w_{n+1}(t) = f(t) + (p-1)\int_t^\infty r^{1-q}(s)f^{q-1}(s)w_n(s)ds.$$

Now

$$\begin{aligned} w_n(t) &= f(t) + (p-1)\int_t^\infty r^{1-q}(s)f^{q-1}(s)w_{n-1}(s)ds \\ &\le f(t) + (p-1)\int_t^\infty r^{1-q}(s)f^{q-1}(s)w_n(s)ds \\ &\le f(t) + (p-1)\int_t^\infty r^{1-q}(s)f^{q-1}(s)w(s)ds \\ &\le w(t), \end{aligned}$$

this implies that $w_n(t) \le w_{n+1}(t) \le w(t)$ for $t \ge T_0$. It is clear from (8) that

$$\int_{T_0}^\infty r^{1-q}(s)f^{q-1}(s)w_{n+1}(s)ds \le \int_{T_0}^\infty r^{1-q}(s)f^{q-1}(s)w(s)ds < \infty.$$

We now see that there is a sequence $\{w_k\}_{k=1}^{\infty}$ satisfying (a), (b), (c), and

$$w_k(t) \le w(t) \tag{9}$$

whenever $k \ge 1$ and $t \ge T_0$. Now (8) and (9) give (e). From (c) we see that the family $\{w_1, w_2, \dots\}$ is equicontinuous, so (9) says that there is a subsequence $\{w_{k_j}\}_{j=1}^{\infty}$ with a locally uniformly limit on $[T_0, \infty)$. This and (b) say that $\{w_k\}_{k=1}^{\infty}$ has a locally uniform limit, say w_0, on $[T_0, \infty)$. Clearly, $w_0 \le w$, so that

$$\int_{T_0}^{\infty} r^{1-q}(s) f^{q-1}(s) w_0(s) ds < \infty.$$

Now, Lebesgue's Dominated Convergence Theorem yields

$$\int_t^{\infty} r^{1-q}(s) f^{q-1}(s) w_0(s) ds = \lim_{k\to\infty} \int_t^{\infty} r^{1-q}(s) f^{q-1}(s) w_k(s) ds$$

for $t \ge T_0$. This implies (d), and (f) is clear from the above discussion, so that the proof is complete.

Corollary 2.3. *If* (E) *is nonoscillatory, then*

$$\limsup_{t\to\infty} \pi^{p-1}(t) \left\{ \int_t^{\infty} c(s) ds + (p-1) \int_t^{\infty} r^{1-q}(s) \left(\int_s^{\infty} c(\tau) d\tau \right)^q ds \right\} \le 1.$$

Proof. As in the proof of Theorem 2.2, we have

$$\limsup_{t\to\infty} \pi^{p-1}(t) w_2(t) \le 1,$$

and

$$\begin{aligned} w_2(t) &= f(t) + (p-1) \int_t^{\infty} r^{1-q}(s) f^{q-1}(s) w_1(s) ds \\ &= f(t) + (p-1) \int_t^{\infty} r^{1-q}(s) f^q(s) ds, \end{aligned}$$

where $f(t) = \int_t^{\infty} c(s) ds$. Hence, the proof is complete.

Corollary 2.4. *If* (E) *is nonoscillatory, then*

$$\int_{t_0}^{\infty} c(s) \exp\left((p-1) \int_{t_0}^{s} r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds < \infty \tag{10}$$

and

$$\int_{t_0}^{\infty} r^{1-q}(s) f^q(s) \exp\left((p-1) \int_{t_0}^{s} r^{1-q}(\tau) f^{q-1}(\tau) d\tau \right) ds < \infty. \tag{11}$$

Proof. As in the proof of Theorem 2.2, there is a number $T_0 \geq t_0$ and a function w_0 on $[T_0, \infty)$ such that

$$w_0(t) = f(t) + (p-1)\int_t^\infty r^{1-q}(s) f^{q-1}(s) w_0(s)ds,$$

where $f(t) = \int_t^\infty c(s)ds$. This implies that

$$w_0'(t) = -c(t) - (p-1) r^{1-q}(t) f^{q-1}(t) w_0(t). \tag{12}$$

Its solution is

$$w_0(T_0) - \int_{T_0}^t c(s)\exp\left((p-1)\int_{T_0}^s r^{1-q}(\tau) f^{q-1}(\tau)d\tau\right)ds$$
$$= w_0(t)\exp\left((p-1)\int_{T_0}^t r^{1-q}(\tau) f^{q-1}(\tau)d\tau\right) > 0.$$

Hence,

$$w_0(T_0) > \int_{T_0}^t c(s)\exp\left((p-1)\int_{T_0}^s r^{1-q}(\tau) f^{q-1}(\tau)d\tau\right)ds.$$

This implies

$$\int_{T_0}^\infty c(s)\exp\left((p-1)\int_{T_0}^s r^{1-q}(\tau) f^{q-1}(\tau)d\tau\right)ds < \infty. \tag{13}$$

Clearly, (13) is equivalent to (10). Let z be given on $[T_0, \infty)$ by

$$z(t) = \int_t^\infty r^{1-q}(s) f^{q-1}(s) w_0(s)ds.$$

Then

$$z'(t) = -r^{1-q}(t) f^{q-1}(t) w_0(t) = -r^{1-q}(t) f^q(t) - (p-1) r^{1-q}(t) f^{q-1}(t) z(t),$$

which implies that (11) holds. Hence, the proof is complete.

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HORNG–JAAN LI
CHIENKUO JUNIOR COLLEGE OF TECHNOLOGY AND COMMERCE
CHANG–HUA
TAIWAN

CHEH–CHIH YEH
DEPARTMENT OF MATHEMATICS
NATIONAL CENTRAL UNIVERSITY
CHUNG–LI
TAIWAN

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