Ideal theory in polynomial semirings

By LOUIS DALE and PAUL J. ALLEN (Alabama)

In this paper, the class of monic ideals will be defined in $R[x]$, where R is a semiring with an identity (all definitions are the same as in ALLEN [1], ALLEN and BRACKIN [2] and BRACKIN [3]). Some methods of constructing monic ideals will be discussed and a characterization of monic ideals in $R[x]$ will be presented. The relationships that exist between monic ideals and k-ideals*) in $Z^+[x]$ (Z^+ denotes the semiring of non-negative integers) will be explored, and particular attention will be devoted to the properties of non-monic k-ideals in $Z^+[x]$.

Definition 1. Let R be a semiring with an identity. An ideal M in $R[x]$ will be called *monic* if $f(x)=\sum a_i x^i \in M$ implies that $a_i x^i \in M$ for $i \in \{0, 1, \ldots, n\}$.

When $\{I_n\}$ is an ascending chain of ideals in the semiring R with identity, the set $\{\sum a_i x^i \in R[x] | a_i \in I_i\}$ will be denoted by I^*.

Theorem 2. *I^* is a monic ideal in $R[x]$.*

PROOF. Let $f(x)=a_n x^n+\ldots+a_0 \in I^*$, $g(x)=b_m x^m+\ldots+b_0 \in I^*$, and $h(x)=c_k x^k+\ldots+c_0 \in R[x]$, where $n \geqq m$. It is clear that $f(x)+g(x)=a_n x^n+\ldots+(a_m+b_m)x^m+\ldots+(a_0+b_0) \in I^*$, since $a_i+b_i \in I_i$ for each i. Consider the product $h(x)f(x)=p_t x^t+\ldots+p_0$, where $p_q=(\sum c_i a_j)x^q$ for $i+j=q$. Since $I_0 \subseteqq I_1 \subseteqq I_2 \subseteqq \ldots$ and $i+j=q$, one has $i \leqq q$ and $c_i \in I_q$ for each i. Similarly, $a_j \in I_q$ for each j. Consequently, $c_i a_j \in I_q$ and it follows that $\sum c_i a_j \in I_q$ for each $q \in \{0, 1, \ldots, n+k\}$. Therefore, $h(x)f(x) \in I^*$ and a similar argument shows $f(x)h(x) \in I^*$. Since $f(x) \in I^*$, it follows that $a_i \in I_i$ and $a_i x^i \in I^*$. Therefore I^* is a monic ideal in $R[x]$.

In view of the fact that ideals in a general semiring are quite numerous, Theorem 2 shows that the class of monic ideals is rather large. In order to show a connection between monic ideals and k-ideals in the semiring $R[x]$, the following theorem is presented.

Theorem 3. *If $\{I_i\}$ is an ascending chain of ideals in the semiring R with an identity, then I^* is a k-ideal in $R[x]$ if and only if each I_i is a k-ideal in R.*

---

*) An ideal I in a semiring R will be called a k-ideal if the following condition is satisfied: If $a \in I$ and $a+b \in I$, then $b \in I$. ([2], 154.)

PROOF. Suppose I^* is formed from the ideals $\{I_i\}$ and each I_i is a k-ideal. Let $f(x)=a_nx^n+\ldots+a_0\in I^*$ and $g(x)=b_mx^m+\ldots+b_0\in R[x]$, where $f(x)+g(x)\in I^*$. It is clear that $a_i\in I_i$ and $a_i+b_i\in I_i$ for each i. However each I_i is a k-ideal and it follows that $b_i\in I_i$. Consequently, $b_ix^i\in I^*$ and it is clear that I^* is a k-ideal. Conversely, suppose that I^* is a k-ideal, $a_i\in I_i$, $b\in R$ and $a_i+b\in I_i$. Since $a_ix^i\in I^*$ and $(a_i+b)x^i= =a_ix^i+bx^i\in I^*$, it is clear that $bx^i\in I^*$. Consequently, $b\in I_i$ and I_i is a k-ideal.

Corollary 4. If A is a k-ideal in a semiring R, then $A[x]$ is a k-ideal in $R[x]$.

The monic ideals displayed thus far have been constructed from ideals in a semiring R. A method of constructing monic ideals using ideals in $R[x]$ will now be considered. When A is an ideal in a semiring $R[x]$, denote the set $\{a\in R|$ there exist $f(x)\in A$ such that a is the coefficient of the ith term of $f(x)\}$ by A_i.

Theorem 5. *If A is an ideal in a semiring $R[x]$, then $\{A_n\}$ is an ascending chain of ideals in R.*

PROOF. If $a\in A_i$ and $b\in A_i$, then there exist polynomials $f(x)\in A$ and $g(x)\in A$ such that a and b are the coefficients of the ith terms of $f(x)$ and $g(x)$, respectively. Since A is an ideal in $R[x]$, it is clear that $f(x)+g(x)\in A$ and $a+b$ is the coefficient of the ith term of $f(x)+g(x)$. Consequently, $a+b\in A_i$. If $c\in R$, then $cf(x)\in A$ and ca is the coefficient of the ith term of $cf(x)$. The same argument shows ac is the ith coefficient of $f(x)c\in A$, and it follows that A_i is an ideal in R for each i. Since $xf(x)\in A$, it follows that a is the coefficient of the $(i+1)$th term of $xf(x)$ and $A_i\subset A_{i+1}$. Consequently, $\{A_n\}$ is an ascending chain of ideals in R.

Definition 6. If A is an ideal in $R[x]$, the ascending chain of ideals $\{A_n\}$ will be called *coefficient ideals.*

When A is an ideal in $R[x]$, one can construct the ideal $A^*=\{\sum a_ix^i\in R[x]|a_i\in A_i\}$ from the ascending chain of coefficient ideals and obtain the following:

Theorem 7. *If A is an ideal in $R[x]$, then A^* is a monic ideal and $A\subset A^*$.*

PROOF. In view of Theorem 5 and Theorem 2, it is clear that A^* is a monic ideal in $R[x]$. If $f(x)=\sum a_ix^i\in A$, then $a_i\in A_i$ for each i and it follows that $a_ix^i\in A^*$. Consequently, $f(x)=\sum a_ix^i\in A^*$ and it has been shown that $A\subset A^*$.

Monic ideals in a semiring $R[x]$ are now characterized by the following:

Theorem 8. *If A is an ideal in $R[x]$, where R is a semiring with an identity, then the following statements are equivalent.*

1. *A is monic.*
2. *$A=A^*$.*
3. *Given $f(x)=a_nx^n+\ldots+a_0\in A$ and $c\in A_i$, the polynomial $f'(x)=a_nx^n+\ldots\ldots+cx^i+\ldots+a_0\in A$.*

PROOF. It is easy to see that $A=A^*$ implies A is monic by Theorem 7. Suppose A is monic and $f(x)=a_nx^n+\ldots+a_0\in A^*$. Since $a_i\in A_i$, there exists polynomials $g_i(x)\in A$ such that a_i is the coefficient of the ith term of $g_i(x)$. The fact that A is monic yields $a_ix^i\in A$ for each $i\in\{0, 1, 2, \ldots, n\}$. Consequently, $f(x)=a_nx^n+\ldots+a_0\in A$ and $A^*\subset A$. In view of Theorem 7, $A\subset A^*$ and it follows that $A=A^*$.

Let A be monic, $f(x)=a_nx^n+\ldots+a_ix^i\ldots+a_0\in A$, and $c\in A_i$. Since $cx^i\in A$ and $a_ix^i\in A$ for each $i\in\{0, 1, \ldots, n\}$, it follows that $f'(x)=a_nx^n+\ldots+cx^i+\ldots+a_0\in A$. Conversely, suppose $f(x)=a_nx^n+\ldots+a_ix^i+\ldots+a_0\in A$ and $c\in A_i$ implies that $f'(x)=a_nx^n+\ldots+c_ix^i+\ldots+a_0\in A$. In view of Theorem 5, A_i is an ideal in R and consequently, $0\in A_i$ for each $i\in\{0, 1, \ldots\}$. Substituting $a_j=0$ for $j\neq i$, the polynomial $f(x)=a_nx^n+\ldots+a_ix^i+\ldots+a_0$ may be reduced to just $a_ix^i\in A$, and the result follows.

It is clear that a monic ideal has a basis constituting of only monomials. The following theorem dictates when that basis is finite.

Theorem 9. *If M is monic in $R[x]$, then M has a finite basis consisting of only monomials if an only if R is Noetherian.*

Proof. Suppose R is Noetherian and M is a monic ideal in $R[x]$. According to Theorem 5, $\{M_n\}$ is an ascending chain of ideals in R. Since R is Noetherian, each M_i has a finite basis, say $\{a_{ij}|j=0, 1, 2, \ldots, m_i\}$. Let N be a positive integer such that $M_i=M_N$ when $i\geqq N$, and let $B=\{a_{ij}x^i|i\in\{0, 1, 2, \ldots, N\}$ and $0\leqq j\leqq m_i\}$. Suppose $f(x)=b_kx^k+\ldots+b_0\in M$. Since $b_i\in M_i$, it follows that $b_i=\sum c_{ij}a_{ij}$ where $c_{ij}\in R$. Consequently, $f(x)=\sum(\sum c_{ij}a_{ij})x^i=\sum(\sum c_{ij}a_{ij}x^i)=\sum(\sum c_{ij}(a_{ij}x^i))$ and B is a finite basis for M. Conversely, suppose every monic ideal in $R[x]$ has a finite basis consisting of only monomials and $\{I_n\}$ is an ascending chain of ideals in R. The ideal I^* formed from this chain is a monic ideal in $R[x]$ and has a finite basis consisting of only monomials, say $D=\{d_{ij}x^i|i=0, 1, 2, \ldots, n$ and $0\leqq j\leqq m_i\}$. Since $d_{ij}\in I_i$ and $\{I_n\}$ is an ascending chain, there exist N such that $d_{ij}\in I_N$ for each i and j. Let $c\in I_m$, where $m\geqq N$. Since $cx^m\in I^*$ and D is a basis for I^*, there exist polynomials $P_{ij}(x)\in R[x]$ such that $cx^m=\sum(\sum P_{ij}(x)d_{ij})x^i$. Since cx^m is a monomial, the expression for cx^m must reduce to $cx^m=(\sum p_{mj}d_{mj})x^m$ where $p_{m_j}\in R$ for each j. Consequently, $c=\sum p_{mj}d_{mj}\in \in I_N$ and $I_m\subset I_N$. Therefore, $I_m=I_N$ and R is Noetherian.

The following theorem gives a relation between monic ideals and k-ideals in a semiring $R[x]$.

Theorem 10. *If A is a k-ideal in $R[x]$, then A monic if and only if A has a basis consisting of only monomials.*

Proof. If the k-ideal A is monic, the remark preceeding Theorem 9 implies A has a basis consisting of only monomials. Conversely, if A has a basis consisting of only monomials say, $B=\{a_mx^{i_m}|m=0, 1, 2, \ldots, n\}$, and $f(x)\in A$, then $f(x)= =f_1(x)a_1x^{i_1}+\ldots+f_n(x)a_nx^{i_n}$ where $f_i(x)\in R[x]$ for each i. When one considers the coefficient of x^i, it is clear that $f(x)=(b_{t_1}a+b_{t_2}a_2+\ldots+b_{t_n}a_n)x^i+f_1'(x)a_1x^{i_1}+\ldots+ +f_n'(x)a_nx^{i_n}$ where b_{t_r} is such that $b_{t_r}x^{i-i_r}$ is the $(i-i_r)$th term of $f_r(x)$ and $f_r'(x)$ is $f_r(x)$ with the $(i-i_r)$th term $b_{t_r}x^{i-i_r}$ removed. In view of the fact that A is a k-ideal and $f_i'(x)a_1x^{i_1}+\ldots+f_n'(x)a_nx^{i_n}\in A$, it follows that $(b_{t_1}a_1+a_{t_1}a_2+\ldots+b_{t_n}a_n)x^i\in A$ and A is monic.

The relationship between k-ideals and monic ideals will now be explored in the semiring $Z^+[x]$. It is an easy matter to show the monic ideal $T_n[x]$, where $T_n= =\{x\in Z^+|x\geqq n\}\cup\{0\}$ and $n>1$, is not a k-ideal in $Z^+[x]$. An example of a k-ideal that is not monic in $Z^+[x]$ will now be given.

Example 11. Consider the semirings $Z^+[x]$ and $Z^+[i]$, where $i=\sqrt{-1}$, and the mapping η of $Z^+[x]$ into $Z^+[i]$ given by $\eta(\sum a_nx^n)=\sum a_ni^n$. If $f(x)=\sum a_nx^n\in Z^+[x]$

and $g(x)=\sum b_n x^n \in Z^+[x]$, then $\eta(f(x)+g(x))=\eta(\sum(a_n+b_n)x^n)=\sum(a_n+b_n)i^n=$ $=\sum(a_n i^n+b_n i^n)=\sum a_n i^n+\sum b_n i^n=\eta(f(x))+\eta(g(x))$. From $f(x)g(x)=(\sum a_n x^n)\cdot$ $\cdot(\sum b_n x^n)=\sum(\sum a_s b_t)x^q$, where $s+t=q$, it follows that $\eta(f(x)g(x))=\sum(\sum a_s b_t)i^q=$ $=(\sum a_n i^n)(\sum b_n i^n)=\eta(f(x))\eta(g(x))$. Consequently, η is a semiring homomorphism and $\ker\eta$ is a k-ideal in $Z^+[x]$. If $a\in Z^+$, then $\eta(a)=(a)i^\circ=a$ and it is clear that $\eta(Z^+)=Z^+$. Thus, theonly constant in $\ker\eta$ is 0. Since $\eta(x^2+1)=i^2+1=0$, it is clear that $x^2+1\in\ker\eta$. However, $1\notin\ker\eta$ since $\eta(1)=1\neq 0$ and it follows that $\ker\eta$ is not a monic ideal.

The following computational lemmas will be essential for exploring non-monic k-ideals in $Z^+[x]$.

Lemma 12. *Let A be a k-ideal in $Z^+[x]$. If $f(x)=a_n x^n+\ldots+a_0\in A$ and v is a non-negative integer, then*

1. $$(a_n x^n+\ldots+a_{i+1}x^{i+1}+a_{i-1}x^{i-1}+\ldots+a_0)^{3^v}+(a_i x^i)^{3^v}\in A \text{ and}$$

2. $$(a_n x^n+\ldots+a_{i+1}x^{i+1}+a_{i-1}x^{i-1}+\ldots+a_0)^{3^v}+(a_i x^i)^{3^v}\notin\langle f(x)\rangle,$$

where $\langle f(x)\rangle$ denotes the ideal generated by $f(x)$.

PROOF. If $h(x)=a_n x^n+\ldots+a_{i+1}x^{i+1}+a_{i-1}x^{i-1}+\ldots+a_0$ and $g(x)=a_i x^i$ then $f(x)=h(x)+g(x)$. Since A is an ideal in $Z^+[x]$, it clear that $[(h(x))^2+(g(x))^2]f(x)\in A$ and $[h(x)g(x)]f(x)\in A$. Thus,

$$\begin{aligned}
[(h(x))^2+(g(x))^2]f(x) &= \\
&= (h(x))^2 f(x)+(g(x))^2 f(x)\\
&= (h(x))^2[h(x)+g(x)]+(g(x))^2[h(x)+g(x)]\\
&= (h(x))^3+(g(x))^3+(h(x))^2 g(x)+(g(x))^2 h(x)\\
&= (h(x))^3+(g(x))^3+[h(x)g(x)][h(x)+g(x)]\\
&= (h(x))^3+(g(x))^3+[h(x)g(x)]f(x).
\end{aligned}$$

Consequently, $(h(x))^3+(g(x))^3=(a_n x^n+\ldots+a_{i+1}x^{i+1}+a_{i-1}x^{i-1}+\ldots+a_0)^3+(a_i x^i)^3\in A$ since A is a k-ideal. In a similar manner it can be shown that $(a_n x^n+\ldots+a_{i+1}x^{i+1}+$ $+a_{i-1}x^{i-1}+\ldots+a_0)^{3^v}+(a_i x^i)^{3^v}\in A$ for $v\in\{0, 1, 2, \ldots\}$. Using integral coefficients, it is easy to see that

$$\begin{aligned}
(h(x))^3+(g(x))^3 &\\
&= [h(x)+g(x)][(h(x))^2-h(x)g(x)+(g(x)_1)^2]\\
&= f(x)[(h(x))^2-h(x)g(x)+(g(x))^2]\notin\langle f(x)\rangle,
\end{aligned}$$

since $[(h(x))^2-h(x)g(x)+(g(x))^2]\notin Z^+[x]$. Similarly, it can be shown that $(h(x))^{3^v}+$ $+(g(x))^{3^v}\notin\langle f(x)\rangle$ for $v\in\{0, 1, 2, \ldots\}$ and the proof is complete.

To find a clue about the structure of non-monic k-ideals in $Z^+[x]$, Example 11 wil be explored further with the aid of the following:

Lemma 13. *If $f(x)\in Z^+[x]$ and $f(i)=0$, then $f(x)=(x^2+1)g_1(x)+\ldots+$ $+(x^{4n+2}+1)g_n(x)$ where $g_i(x)\in Z^+[x]$ for each* i.

PROOF. Without loss of generality, assume the degree of $f(x)$ is even and $f(x)=f_1(x)+f_2(x)$ where $f_1(x)=a_{2t}x^{2t}+a_{2t-2}x^{2t-2}+\dots+a_2x^2+a_0$ and $f_2(x)=$ $=a_{2t-1}x^{2t-1}+a_{2t-3}x^{2t-3}+\dots+a_3x^3+a_1x^1$. Since $f(i)=0$, it is clear that $f_1(i)=$ $=f_2(i)=0$. If $f_1(x)$ has p terms and $a_{2t-2}\neq 0$, then either (1) $a_{2t}=a_{2t-2}$, (2) $a_{2t}<a_{2t-2}$ or (3) $a_{2t}>a_{2t-2}$. If $a_{2t}=a_{2t-2}$, then $a_{2t}x^{2t}+a_{2t-2}x^{2t-2}=a_{2t}x^{2t-2}(x^2+1)$. If $a_{2t}<a_{2t-2}$ then there exists $a'_{2t-2}\in Z^+$ such that $a_{2t}+a'_{2t-2}=a_{2t-2}$ and consequently, $a_{2t}x^{2t}+$ $+a_{2t-2}x^{2t}=a_{2t}x^{2t-2}(x^2+1)+a'_{2t-2}x^{2t-2}$. If $a_{2t}>a_{2t-2}$, a similar argument shows $a_{2t}x^{2t}+a_{2t-2}x^{2t-2}=a'_{2t}x^{2t}+a_{2t-2}x^{2t-2}(x^2+1)$. In either case, $f_1(x)=(x^{4n_1+2}+1)g_{11}(x)+$ $+R_1(x)$ where $g_{11}(x)\in Z^+[x]$ and $R_1(x)$ has at most $p-1$ terms. If $a_{2t-2}=0$, it is clear that the same result can be obtained using a_{2t-4}. If $R_1(x)=0$, the proof is complete. When $R_1(x)\neq 0$, there are three possibilities for each of the above three cases. An example of one of nine new cases follows: if $a_{2t}>a_{2t-2}$ and $a'_{2t}<a_{2t-6}$, then there exists $a'_{2t-4}\in Z^+$ such that $a'_{2t}+a'_{2t-6}=a_{2t-6}$ and consequently $a_{2t}x^{2t}+$ $+a_{2t-2}x^{2t-2}+a_{2t-6}x^{2t-6}=(x^2+1)a_{2t-2}x^{2t-2}+(x^6+1)a'_{2t}x^{2t-6}+a_2'^{t-6}x^{2t-6}$. It can be shown that similar expressions exist in the other eight cases, and consequently, $f_1(x)=(x^{4n_1+2}+1)g_{11}(x)+(x^{4n_2+2}+1)g_{12}(x)+R_2(x)$ where $g_{1i}(x)$, $R_2(x)\in Z^+[x]$ and $R_2(x)$ has at most $p-2$ terms. Since p is finite and $f_1(i)=0$, it is clear that this procedure can be continued only a finite number of steps to obtain $f_1(x)=(x^{4n_1+2}+1)g_{11}(x)+\dots$ $\dots+(x^{4n_s+2}+1)g_{1s}(x)$ where $g_{1i}(x)\in Z^+[x]$. Let $n=\max\{n_1, n_2, \dots, n_s\}$. After re-indexing and considering the possibility that some $g_{1i}(x)$ may equal 0, one has $f_1(x)=(x^2+1)g_{11}(x)+(x^6+1)g_{12}(x)+\dots+(x^{4n+2}+1)g_{1n}(x)$. In a similar manner it can be shown that $f_2(x)=(x^2+1)g_{21}(x)+(x^6+1)g_{22}(x)+\dots+(x^{4n+2}+1)g_{2n}(x)$. Consequently, $f(x)=(x^2+1)g_1(x)+(x^6+1)g_2(x)+\dots+(x^{4n+2}+1)g_n(x)$ where $g_i(x)=$ $=g_{1i}(x)+g_{2i}(x)\in Z^+$ for each i.

Returning to Example 11, let $A_0=\langle x^2+1\rangle$, $A_1=\langle x^6+1\rangle+A_0, \dots, A_n=\langle x^{4n+2}+1\rangle+$ $+A_{n-1}, \dots$ and let $A=\bigcup A_i$. It is easy to see that $\{A_i\}$ is a proper ascending chain of ideals in $Z^+[x]$. Assume A_n is a k-ideal in $Z^+[x]$. Lemma 12 implies $(x^{4n+2})^3+1\in A_n$, since $x^{4n+2}+1\in A_n$. However, it is easy to show that $(x^{4n+2})^3+1\notin A_n$, a contradiction. Therefore, A is the union of a proper ascending chain of non k-ideals. If $f(x)\in A$, then there exists $p\in Z^+$ such that $f(x)\in A_p$ and it follows that $f(x)=$ $=(x^2+1)f_0(x)+(x^6+1)f_1(x)+\dots+(x^{4p+2}+1)f_p(x)$. Since

$$\begin{aligned}\eta f(x) &= \eta(n+1)\eta\big(f_0(x)\big)+\eta(x^6+1)\eta f_1(x)+\dots+\eta(x^{4p+2}+1)\eta f_p(x)\\ &= (i^2+1)f_0(i)+(i^6+1)f_1(i)+\dots+(i^{4p+2}+1)f_p(i)\\ &= 0\cdot f_0(i)+0\cdot f_1(i)+\dots+0\cdot f_p(i) = 0,\end{aligned}$$

it follows that $f(x)\in\ker\eta$ and consequently, $A\subset\ker\eta$. On the other hand, if $f(x)\in\ker\eta$, then $\eta\big(f(x)\big)=f(i)=0$ by Lemma 13

$$f(x) = (x^2+1)g_1(x)+(x^6+1)g_2(x)+\dots+(x^{4n+2}+1)g_n(x).$$

Therefore, $\ker\eta\subset A$ and it follows that $\ker\eta=A$. This shows that the example of a non-monic k-ideal considered in Example 11 is the union of a proper ascending chain of non k-ideals.

Theorem 14. *If A is an ideal in $Z^+[x]$ with a finite basis B, where B does not contain any monomials, then A is not a k-ideal.*

PROOF. Assume that A is a k-ideal and $B=\{g_1(x), \ldots, g(x)\}$ is a finite basis for A where B does not contain any monomials. If $f(x)=a_n x^n+a_{n-p}x^{n-p}+\ldots+a_0 \in Z^+[x]$ where $a_i=0$ for $n-1 \geqq i \geqq n-p+1$, define a function S mapping $Z^+[x]$ onto Z^+ by $S(f(x))=n-(n-p)=p$. Suppose $S(g_i(x))=c_i$ and $c=\max\{c_1, c_2, \ldots, c_n\}$. When $f(x)\in A$ and $S(f(x))=p$, Lemma 12 implies $f_\tau(x)=(a_n x^n)^{3^\tau}+(a_{n-p}x^{n-p}+\ldots+a_0)^{3^\tau}\in A$ for $\tau\in\{0, 1, 2, \ldots\}$. Since p is fixed and $S(f_\tau(x))=3^\tau p$, it is clear that $\{3^\tau p\}$ is an increasing sequence of positive integers. Consequently, there exists a τ such that $3^\tau p>c$. It is clear that

$$(1)\qquad f_\tau(x) = (a_n x^n)^{3^\tau}+(a_{n-p}x^{n-p}+\ldots+a_0)^{3^\tau} = h_1(x)g_1(x)+h_2(x)g_2(x)+\ldots+h_n(x)g_n(x),$$

since $f_\tau(x)\in A$. It is clear that $S(f_\tau(x))=3^\tau p$. Since $(a_n x^n)^{3^\tau}$ appears on the left hand side of equation (1), at least one of the products, say $h_i(x)g_i(x)$, must produce a term of degree $3^\tau n$. It follows that $g_i(x)=b_m x^m+b_{m-c_i}x^{m-c_i}+\ldots+b_0$, since $S(g_i(x))=c_i$. Moreover, $h_i(x)$ must have a term of the form $px^{3^\tau n-m}$ and $px^{3^\tau n-m}g_i(x)=pb_m x^{3^\tau n}+pb_{m-c_i}x^{3^\tau-c_i}+\ldots+pb_0 x^{3^\tau n-m}$ is a part of the product $h_i(x)g_i(x)$. Consequently, the right hand side of equation (1) contains a term of degree $3^\tau n-c_i$. Since $c=\max\{c_1, c_2, \ldots, c_n\}$ and $3^\tau p>c$, it follows that (2) $3^\tau n>3^\tau n-c_i>3^\tau n-c>3^\tau n-3^\tau p=3^\tau(n-p)$. The second highest degree term appearing on the left hand side of (1) is of degree $3^\tau(n-p)$, since $S(f_\tau(x))=3^\tau p$. However, a term of degree $3^\tau n-c_i$ cannot appear on the left hand side of (1) because of (2), a contradiction.

Corollary 15. If A is a k-ideal in $Z^+[x]$ and A does not contain any monomials, then every basis for A is infinite.

The results arising from Example 11 will be generalized in the following:

Theorem 16. *Let A be a k-ideal in $Z^+[x]$. If A does not contain any monomials, then $A=\cup A_i$ where $\{A_i\}$ is a proper ascending chain of ideals and A_i is not a k-ideal for each $i\in Z^+$.*

PROOF. It follows from Corollary 15 that A has an infinite basis, say $B=\{g_0(x), g_1(x), \ldots, g_n(x), \ldots\}$. Let $A_0=\langle g_0(x)\rangle$, $A_1=\langle g_1(x)\rangle+A_0, \ldots, A_n=\langle g_n(x)\rangle+A_{n-1}, \ldots$. Corollary 15 implies $\{A_i\}$ contains an infinite number of distinct ideals, and Theorem 14 guarantees that A_i is not a k-ideal for each i and the result follows.

It has been shown that a k-ideal in $Z^+[x]$ is monic if and only if it has a basis consisting of only monomials. It has also been shown that a k-ideal that does not contain any monomials can not have a finite basis. In the first case the basis consisted of only monomials, while in the second case the basis was void of monomials. It has been shown that an ideal with a finite basis void of monomials is not a k-ideal.

It may appear at first glance that an ideal having a "mixed", irredundant, finite basis is not a k-ideal. However the following two examples will show that an ideal in $Z^+[x]$ having a "mixed", irredundant, finite basis, may or may not be a k-ideal.

Example 17. Let $A=\langle f(x)\rangle$, where $f(x)=2x^4+4x^2$, let $B=\langle 8x^2, 4x^3, 2x^5\rangle$ be the ideal generated by $\{8x^2, 4x^3, 2x^5\}$ and suppose $C=A+B$. It can be shown that B is a monic k-ideal in $Z^+[x]$. In order to show that C is a non-monic k-ideal, the following four properties of C will be needed.

Property 1. Let $h(x)=a_n x^n+\dots+a_0\in Z^+[x]$. If $a_0\neq 1$, then $h(x)f(x)\in B$. If $a_0=1$, then $h(x)f(x)=f(x)+h'(x)$ where $h'(x)\in B$.

PROOF. Since $xf(x)=2x^5+4x^3\in B$ and $pf(x)=2px^4+4px^2\in B$ when $p>1$, it follows that $a_i x^i f(x)\in B$ and $h(x)f(x)\in B$ if $a_0\neq 1$. If $a_0=1$, then

$$\begin{aligned} h(x)f(x) &= (a_n x^n+\dots+a_1 x+1)f(x)\\ &= (a_n x^n+\dots+a_1 x)f(x)+f(x)\\ &= h'(x)+f(x) \text{ where } h'(x)\in B. \end{aligned}$$

Property 2. If $g(x)\in C$, then $g(x)\in B$ or $g(x)=f(x)+g'(x)$ where $g'(x)\in B$.

PROOF. If $g(x)\in C$, then $g(x)=h(x)f(x)+k(x)$ where $k(x)\in B$. Letting $h(x)=$ $=a_n x^n+\dots+a_0$, it follows from Property 1 that $h(x)f(x)\in B$ or $h(x)f(x)=f(x)+h'(x)$ where $h'(x)\in B$. Consequently, $g(x)=h(x)f(x)+k(x)\in B$ or $g(x)=h(x)f(x)+k(x)=$ $=f(x)+h'(x)+k(x)=f(x)+g'(x)$ where $g'(x)=h'(x)+k(x)\in B$.

Property 3. If $g(x)\in B$, where $g(x)\neq 0$, then $g(x)+f(x)\notin B$.

PROOF. Assume $g(x)+f(x)\in B$. Since B is a k-ideal in $Z^+[x]$, it follows that $f(x)\in B$, a contradiction.

Property 4. If $g(x)+f(x)\in B$, then $g(x)=f(x)+g'(x)$ where $g'(x)\in B$.

PROOF. Let $g(x)=a_n x^n+\dots+a_4 x^4+a_3 x^3+a_2 x^2$ and suppose $g(x)+f(x)\in B$. Since $g(x)+f(x)=a_n x^n+\dots+(a_4+2)x^4+a_3x^3+(a_2+4)x^2$ it follows that $a_4+2\in B_4=(4)$ and $a_2+4\in B_2=(8)$, where B_4 and B_2 are coefficient ideals of B. Consequently, $a_4=4k+2$ and $a_2=8m+4$ for some $k\in Z^+$ and $m\in Z^+$. Therefore,

$$\begin{aligned} g(x) &= a_n x^n+\dots+(4k+2)x^4+a_3x^3+(8m+4)x^2 = \\ &= a_n x^n+\dots+4kx^4+a_3x^3+8mx^2+2x^4+4x^2 = g'(x)+f(x). \end{aligned}$$

Since B is monic, it can be shown that $g'(x)\in B$.

Suppose $g(x)\in C$ and $h(x)\in Z^+[x]$ such that $h(x)+g(x)\in C$. It follows from Property 2 that $g(x)\in B$ or $g(x)=f(x)+g'(x)$ where $g'(x)\in B$, and that $h(x)=g(x)\in B$ or $h(x)+g(x)=f(x)+k(x)$ where $k(x)\in B$.

Case 1. If $g(x)\in B$ and $g(x)+h(x)\in B$, then $h(x)\in B$, since B is a k-ideal.

Case 2. If $g(x)\in B$ and $g(x)+h(x)=f(x)+k(x)$, then it follows from Property 3, that $h(x)\notin B$. Consequently, $h(x)=f(x)+h'(x)$ where $h'(x)\in B$ and it follows that $h(x)\in C$.

Case 3. If $g(x)=f(x)+g'(x)$ and $g(x)+h(x)\in B$, then $g(x)+h(x)=f(x)+g'(x)+$ $+h(x)$ and since B is a k-ideal, it is clear that $f(x)+h(x)\in B$. Consequently, by Property 4, $h(x)=f(x)+h'(x)$ where $h'(x)\in B$ and it follows that $h(x)\in C$.

Case 4. If $g(x)=f(x)+g'(x)$ and $g(x)+h(x)=f(x)+k(x)$, then $f(x)+g'(x)+$ $+h(x)=f(x)+k(x)$ and consequently $g'(x)+h(x)+k(x)\in B$. Since B is a k-ideal, it follows that $h(x)\in B\subset C$.

In either case $h(x)\in C$ and C is a k-ideal. It is easy to see that C is not a monic ideal. Therefore C is a non-monic k-ideal with a finite basis.

Example 21. Let $A=\langle 2x^4+4x^2\rangle$, $B=\langle 8x^2\rangle$, and $C=A+B$. Since $2(2x^4+4x^2)=$ $=4x^4+8x^2\in C$ and $8x^2\in C$, but $4x^4\notin C$, it follows that C is not a k-ideal.

Bibliography

[1] P. J. ALLEN, Operator semigroups with applications to semirings, *Publ. Math. (Debrecen)* **20** (1973), 161—175.
[2] E. J. BRACKIN and P. J. ALLEN, A basis theorem for the semiring part of a Boolean algebra, *Publ. Math. (Debrecen)* **20** (1973), 153—155.
[3] E. J. BRACKIN, Semirings and the Hilbert Basis Theorem, *Dissertation of University of Alabama.*

(Received March 10, 1974.)