A new algebra of distributions; initial value problems involving Schwartz distributions II.

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§ 3. The operational calculus

Let $\mathscr{C}^\infty$ be the linear space of all the complex-valued functions which are infinitely differentiable on $\mathbf{R}$. Further, let W be the linear space of all the functions $w(\,)$ in $\mathscr{C}^\infty$ such that $w^{(k)}(0)=0$ for each integer $k\geqq 0$. For example, if $q(\,)$ is the function defined by $q(0)=0$ and

$$q(x) = \exp\left(\frac{-1}{|x|}\right) \quad (\text{for } x\neq 0), \tag{3.1}$$

then

$$q(\,) \text{ belongs to } W. \tag{3.2}$$

An *operator* is a linear mapping of W into W. If A is an operator and if $w(\,)\in W$, we denote by $.Aw(\,)$ the function that the operator A assigns to $w(\,)$.

3.3. *The space* $\mathscr{A}$. Let $\mathscr{A}$ be the space (denoted $\mathscr{A}_\omega$ in [7]) of all the operators A such that the equation

$$.A(w_1\wedge w_2)(\,) = (.Aw_1)\wedge w_2(\,) \tag{3.4}$$

holds whenever $w_1(\,)$ and $w_2(\,)$ belong to W. The operation $\wedge$ was defined in 2.33.

The linear space $\mathscr{A}$ is a subalgebra of the algebra of operators, multiplication being defined in the usual way: the product A_1A_2 of two operators is determined by the equation

$$.A_1A_2w(\,) = .A_1(.A_2w)(\,) \qquad (\text{for } w(\,) \text{ in } W).$$

In fact $\mathscr{A}$ is a commutative algebra [6, 1.38] and $\mathscr{A}$ contains the differentiation operator D defined by

$$.Dw(\,) = w'(\,) \qquad (\text{for } w(\,) \text{ in } W): \tag{3.5}$$

see [6, 1.30]. Let I be the identity-operator defined by $.Iw(\,)=w(\,)$ for every $w(\,)$ in W: the operator I is the multiplicative unit of the algebra $\mathscr{A}$.

If $f(\,)\in\mathscr{F}$ we denote by f the mapping defined by

$$.fw(\,) = f\wedge w'(\,) \qquad (\text{for } w(\,) \text{ in } W); \tag{3.6}$$

it can be proved that f is an operator, called *the operator of the function* $f(\ )$; in fact, $f\in\mathscr{A}$ (see 1.34 in [6]). The identity-operator I is the operator 1 of the unit constant function $1(\ )$ defined by $1(t)=1$ for $t\in\mathbf{R}$ (see [6, 1.29]); thus,

$$1A = A = A1 \qquad \text{(for } A \text{ in } \mathscr{A}). \tag{3.7}$$

Moreover, the transformation $f(\ )\mapsto f$ is a linear injection of $\mathscr{F}$ into $\mathscr{A}$ (see 1.34 in [6]).

3.8. *Reorientation.* We shall define a linear injection $R\mapsto R^1$ of $\mathfrak{B}$ into $\mathscr{A}$ such that

$$[f]^{01} = f \qquad \text{(when } f(\ )\in\mathscr{F}), \tag{3.9}$$

$$[F\otimes S]^1 = F^1D^{-1}S^1 \qquad \text{(for } F \text{ and } S \text{ in } \mathfrak{B}), \tag{3.10}$$

$$([1]^0\otimes S)^1 = D^{-1}S^1 \qquad \text{(for } S \text{ in } \mathfrak{B}), \tag{3.11}$$

and

$$\delta^1 = D1_+. \tag{3.12}$$

We begin with some preliminary definitions and lemmas.

3,13. **Lemma.** *If* $w(\ )\in W$ *then both* $w_-(\ )$ *and* $w_+(\ )$ *belong to* $\mathscr{C}^\infty$.

Proof. Let k be any integer $\geqq 1$. Since

$$w^{(k)}(0) = \lim_{t\to 0}\frac{w^{(k-1)}(t)-w^{(k-1)}(0)}{t-0},$$

the equation

$$0 = \lim_{t\to 0}\frac{w^{(k-1)}(t)}{t} \tag{2}$$

is an immediate consequence of the fact that $w^{(n)}(0)=0$ for each integer $n\geqq 0$. Set $f_1(\ )=w_-(\ )$ and $f_2(\ )=w_+(\ )$ (see 2.34—2.35). Suppose that $1\leqq m\leqq 2$: we must prove that $f_m^{(k)}(t)$ exists for any t in $\mathbf{R}$. Since $f_m^{(k)}(t)$ exists for $t\neq 0$, it will suffice to prove that $f_m^{(k)}(0)=0$. Since $f_m^{(0)}(0)=0$, we proceed by induction on k. To that effect, assume that $f_m^{(k-1)}(0)=0$, which implies that

$$f_m^{(k)}(0) = \lim_{t\to 0}\frac{f_m^{(k-1)}(t)-0}{t} = 0: \tag{3}$$

the second equation is obtained by noting that $f_m^{(k-1)}(t)/t$ either equals 0 or it equals $w^{(k-1)}(t)/t$ (and using (2)). This completes the induction proof of the equation $f_m^{(k)}(0)=0$ for each integer k.

3.14. Definition. *Let* $\mathscr{C}_-^\infty$ *(respectively,* $\mathscr{C}_+^\infty$*) be the linear space of all the functions in* $\mathscr{C}^\infty$ *which vanish on* $[0,\infty)$ *(respectively, on* $(-\infty,0]$*).*

3.15. **Lemma.** *If* $w(\ )\in W$ *then* $w_-(\ )\in\mathscr{C}_-^\infty$ *and* $w_+(\ )\mathscr{C}_+^\infty$*; moreover,* $W=\mathscr{C}_-^\infty+\mathscr{C}_+^\infty$.

PROOF. The two properties $w_-()\in\mathscr{C}_-^\infty$ and $w_+()\in\mathscr{C}_+^\infty$ are immediate from 3.13 and 2.34—2.35. Since both $\mathscr{C}_-^\infty$ and $\mathscr{C}_+^\infty$ are subsets of W, we only need to prove that W is a subset of $\mathscr{C}_-^\infty+\mathscr{C}_+^\infty$. To that effect, take $w()$ in W and note that $w()=w_-()+$ $+w_+()$ (by 2.34—2.35): the conclusion $w()\in\mathscr{C}_-^\infty+\mathscr{C}_+^\infty$ is now immediate from the fact that $w_-()\in\mathscr{C}_-^\infty$ and $w_+()\in\mathscr{C}_+^\infty$.

3.16. **Lemma.** *There is a bilinear mapping*

$$(3.17)\qquad \big(\alpha(\,),\,T\big)\mapsto \alpha\,\Box\, T(\,)\in\mathscr{C}_+^\infty$$

of the cartesian product $\mathscr{C}_+^\infty\times\mathfrak{B}_+$ *into* $\mathscr{C}_+^\infty$ *such that*

$$(3.18)\qquad [\alpha\,\Box\, T]^0=[\alpha]^0 * T.$$

PROOF. Suppose that $\alpha()\in\mathscr{C}_+^\infty$. If $x\in\mathbf{R}$, let $\alpha_x()$ be the function defined by $\alpha_x(t)=\alpha(x-t)$ (for $t\in\mathbf{R}$). If $T\in\mathfrak{B}_+$ we denote by $\alpha\,\Box\, T()$ the function defined by

$$(4)\qquad \alpha\,\Box\, T(x)=T(\alpha_x)\qquad (\text{for } x\in\mathbf{R}).$$

Since the supports of $[\alpha]^0$ and T are both contained in $[0,\infty)$, these distributions satisfy condition $(\sum)$ in [4, p. 383]; we may therefore apply Proposition 1 in [4, p. 402] with $s=\infty=t$: what is there called E^s and E^{s-t} becomes $\mathscr{C}^\infty$ (see [4, p. 440]); further, D'^s becomes D'; noting that $\alpha * T$ is what we denote by $[\alpha]^0 * T$, Proposition 1 states that $[\alpha]^0 * T(x)$ equals the righthand side of (4): consequently, $[\alpha\,\Box\, T]^0=[\alpha]^0 * T$. Having thus verified 3.18, we can use the second assertion in Proposition 1 [loc cit.] to state that

$$(5)\qquad \alpha\,\Box\, T(\,)\text{ belongs to }\mathscr{C}^\infty.$$

Since both $\mathbf{O}([\alpha]^0)$ and $\mathbf{O}(T)$ contain the interval $(-\infty,0)$, it follows from 2.8 that

$$\mathbf{O}([\alpha]^0 * T)\supset(-\infty,0),$$

whence the relation $\mathbf{O}(\alpha\,\Box\, T)\supset(-\infty,0)$ now follows from 3.18: combining with (5), we conclude that $\alpha\,\Box\, T()$ belongs to $\mathscr{C}_+^\infty$.

In consequence, we have established that the mapping 3.17 is into $\mathscr{C}_+^\infty$; it is readily inferred from 3.18 that it is a bilinear mapping.

3.19. **Theorem.** *There is a bilinear mapping*

$$(3.20)\qquad \big(R,\,w(\,)\big)\mapsto R\triangle w(\,)\in W$$

of the cartesian product $\mathfrak{B}\times W$ *into* W *such that*

$$(3.21)\qquad [R\triangle w]^0=R\otimes[w]^0\qquad \big(\textit{for } w(\,) \textit{ in } W\big),$$

and

$$(3.22)\qquad R\triangle q(\,)=0\quad \textit{implies}\quad R=\mathbf{0},$$

where $q()$ *is the function defined by* 3.1.

PROOF. Suppose that $R \in \mathfrak{B}$. If $w(\,) \in W$ it follows from 3.13 that $w_+(\,) \in \mathscr{C}_+^\infty$; we may therefore set $\alpha = w_+$ in 3.16 to infer that

$$(1) \qquad w_+ \,\square\, R_+(\,) \text{ belongs to } \mathscr{C}_+^\infty \qquad (\text{since } R_+ \in \mathfrak{B}_+),$$

and

$$(2) \qquad [w_+ \,\square\, R_+]^0 = R_+ * [w_+]^0:$$

the last equation is from 3.18 and 2.9.

Since $w_+^\vee(x) = w_-(-x) = 0$ when $-x > 0$, we see that $w_-^\vee(\,)$ vanishes on $(-\infty, 0)$: this function $w_-^\vee(\,)$ therefore belongs to $\mathscr{C}_+^\infty$. On the other hand, $R_- \in \mathfrak{B}_-$, so that $\mathbf{O}(R_-) \supset (0, \infty)$, whence

$$\mathbf{O}\,(R)_-^\vee \supset (-\infty, 0) \qquad (\text{by } 2.6).$$

Thus, $w_-^\vee(\,) \in \mathscr{C}_+^\infty$ and $R_-^\vee \in \mathfrak{B}_+$: from 3.16 it therefore follows that

$$(3) \qquad w_-^\vee \,\square\, R_-^\vee(\,) \quad \text{belongs to} \quad \mathscr{C}_+^\infty$$

and

$$(4) \qquad [w_-^\vee \,\square\, R_-^\vee]^0 = R_-^\vee * [w]^0:$$

the last equation is from 3.18 and 2.9. From (3) it follows easily that

$$(5) \qquad (w_-^\vee \,\square\, R_-^\vee)^\vee \quad \text{belongs to } \mathscr{C}_-^\infty.$$

We now define the function $R \triangle w(\,)$ by the equation

$$(6) \qquad R \triangle w(\,) = -(w_-^\vee \,\square\, R_-^\vee)^\vee(\,) + w_+ \,\square\, R_+(\,).$$

From (1), (5), (6), and 3.15 it follows that $R \triangle w(\,)$ belongs to W; it is now easily verified that the mapping 3.21 is bilinear. It remains to prove 3.21—3.22. In view of 2.3, Equation (6) implies that

$$[R \triangle w]^0 = -[w_-^\vee \,\square\, R_-^\vee]^{0\vee} + [w_+ \,\square\, R_+]^0,$$

so that (4) and (2) give

$$(7) \qquad [R \triangle w]^0 = -(R_-^\vee * [w_-^\vee]^0)^\vee + R_+ * [w_+]^0.$$

From (7), 2.3, and 2.12 we see that

$$[R \triangle w]^0 = -R_- * [w_-]^0 + R_+ * [w_+]^0:$$

Conclusion 3.21 now comes from 2.18 and 1.28. Finally, suppose that $R \triangle q(\,) = 0$; therefore,

$$(8) \qquad 0 = (R \triangle q)_+(\,) = q_+ \,\square\, R_+(\,)$$

and

$$(9) \qquad 0 = (R \triangle q)_-^\vee(\,) = (q_-^\vee \,\square\, R_-^\vee)(\,):$$

the right-hand equations are immediate from (6), (5), and (1)—(2). From (8)—(9) and 3.18 we see that

$$(10) \qquad [q_+]^0 * R_+ = \mathbf{0} = [q_-^\vee]^0 * R_-^\vee;$$

but $[q_+]^0$ and $[q_-^{\smile}]^0$ belong to the Schwartz space D'_+, which has the property that $A*T=\mathbf{0}$ implies $T=\mathbf{0}$ whenever $A\neq\mathbf{0}$ (see [8, pp. 172—173]); since $[q_+]^0\neq\mathbf{0}$ and $[q_-^{\smile}]^0\neq\mathbf{0}$, the conclusion $R_+=\mathbf{0}=R_-^{\smile}$ now come from (10): since $R_-^{\smile}=0$ implies $R_-=0$ (by 2.4), we have $R_-=\mathbf{0}=R_+$, whence our conclusion $R=\mathbf{0}$.

3.23. Definition. *If $R\in\mathfrak{B}$ we denote by $R^\triangle$ the mapping that assigns to each $w(\,)$ in W the function $R\triangle w(\,)$.*

3.24. *Remark.* If $w(\,)\in W$ then $R\triangle w\in W$ (by 3.20); therefore, $R^\triangle$ is the operator defined by

$$.R^\triangle w(\,) = R\triangle w(\,) \qquad \text{(for } w(\,) \text{ in } W). \tag{3.25}$$

From 3.21 we see that

$$[.R^\triangle w]^0 = R\otimes[w]^0 \qquad \text{(for } w(\,) \text{ in } W). \tag{3.26}$$

3.27. **Lemma.** *If $R\in\mathfrak{B}$ then $R^\triangle\in\mathscr{A}$.*

PROOF. In view of 3.24, it only remains to verify that 3.4 holds in case $A=R^\triangle$. The equations

$$[.R^\triangle(w_1\wedge w_2)]^0 = R\otimes[w_1\wedge w_2]^0 = R\otimes([w_1]^0\otimes[w_2]^0) \tag{1}$$

are from 3.26 and 2.32. From (1) and 2.28 it follows that

$$[.R^\triangle(w_1\wedge w_2)]^0 = (R\otimes[w_1]^0)\otimes[w_2]^0 = [.R^\triangle w_1]^0\otimes[w_2]^0: \tag{2}$$

the last equation is from 3.26. From (2) and 2.32 we see that

$$[.R^\triangle(w_1\wedge w_2)]^0 = [(.R^\triangle w_1)\wedge w_2]^0;$$

in view of 1.11. the conclusion

$$.R^\triangle(w_1\wedge w_2)(\,) = (.R^\triangle w_1)\wedge w_2(\,)$$

is now at hand.

3.28. **Lemma.** *If F and S belong to $\mathfrak{B}$, then*

$$(F\otimes S)^\triangle = F^\triangle S^\triangle. \tag{3.29}$$

PROOF. For any $w(\,)$ in W it follows from 3.26 that

$$[.(F\otimes S)^\triangle w]^0 = (F\otimes S)\otimes[w]^0 = F\otimes(S\otimes[w]^0)$$

consequently, two more applications of 3.26 give

$$[.(F\otimes S)^\triangle w]^0 = F\otimes[.S^\triangle w]^0 = [.F^\triangle(.S^\triangle w)]^0. \tag{3}$$

From (3) and 1.11 we therefore have

$$.(F\otimes S)^\triangle w(\,) = .F^\triangle(.S^\triangle w)(\,) = .(F^\triangle S^\triangle)w(\,):$$

the last equation is from the definition of multiplication of operators. Since $w(\,)$ is an arbitrary element of W, the proof is completed.

3.30. Definition. *If $R\in\mathfrak{B}$, we set $R^1=R^\triangle D$.*

3.31. *Remark*. Since D and $R^{\triangle}$ belong to the algebra $\mathscr{A}$ (see 3.3 and 3.27), we see that $R^1 \in \mathscr{A}$. Let us verify that

$$[.R^1 w]^0 = R \otimes [w']^0 \qquad \text{(for } w(\,) \in W\text{)}: \tag{3.32}$$

indeed, the equations

$$[.R^1 w]^0 = [.R^{\triangle}(.Dw)]^0 = [.R^{\triangle} w']^0 = R \otimes [w']^0$$

are from 3.30 and 3.26.

3.33. **Theorem.** *If F and S belong to $\mathfrak{B}$, then*

$$[F \otimes S]^1 = F^1 D^{-1} S^1. \tag{3.34}$$

Proof. The equations

$$[F \otimes S]^1 = [F \otimes S]^{\triangle} D = (F^{\triangle} S^{\triangle}) D = (F^{\triangle} D) D^{-1} (S^{\triangle} D)$$

are from 3.30, 3.29, and the associativity of the algebra $\mathscr{A}$; another application of 3.30 now gives 3.34.

3.35. **Theorem.** *If $f(\,) \in \mathscr{F}$ then*

$$[f]^{01} = f. \tag{3.36}$$

Proof. If $w(\,) \in W$ the equations

$$[.[f]^{01} w]^0 = [f]^0 \otimes [w']^0 = [f \wedge w']^0 = [.fw]^0$$

are from 3.32, 2.32, and 3.6. From 1.11 it therefore follows that $.[f]^{01} w(\,) = .fw(\,)$; since $w(\,)$ is arbitrary, Conclusion 3.36 is at hand.

3.37. *Remarks*. If $f(\,) \in \mathscr{F}$ we can combine 3.36 with 3.34 to obtain

$$([f]^0 \otimes S)^1 = f D^{-1} S^1 \qquad \text{(if } S \in \mathfrak{B}\text{)}. \tag{3.38}$$

Setting $f(\,) = 1(\,)$ in 3.38, we obtain

$$([1]^0 \otimes S)^1 = 1 D^{-1} S^1 = D^{-1} S^1: \tag{3.39}$$

the second equation is from 3.7. Setting $S = \delta$ in 3.39, we obtain $([1]^0 \otimes \delta)^1 = D^{-1} \delta^1$; consequently, the equations

$$\delta^1 = D([1]^0 \otimes \delta)^1 = D(\delta \otimes [1]^0)^1 = D[1]^{01}_{+} \tag{4}$$

are from 2.27 and 2.29. From (4), 1.28, and 3.36 it follows that

$$\delta^1 = D 1_{+}. \tag{3.40}$$

3.41. **Theorem.** *The transformation $R \to R^1$ is a linear injection of $\mathfrak{B}$ into $\mathscr{A}$. In particular,*

$$T^1 = S^1 \quad \textit{implies } T = S. \tag{3.42}$$

PROOF. The linearity follows directly from Definitions 3.30 and 3.25 (the bilinearity property is stated in 3.19); the fact that $R^1 \in \mathscr{A}$ was verified in 3.31. To prove 3.42, set $R = T - S$: the hypothesis $R^1 = \mathbf{0}$ implies $R^\triangle D = \mathbf{0}$ (by 3.30); right-multiplying by D^{-1} both sides of this equation, we obtain $R^\triangle = \mathbf{0}$, whence $.R^\triangle q(\,) = 0$ (since $q(\,) \in W$: see 3.2), whence $R^\triangle q(\,) = 0$ (by 3.25), whence the conclusion $R = \mathbf{0}$ now comes from 3.22.

3.43. *Reorientation.* We have now proved all the properties announced in 3.9—3.12. We now prepare for § 4. If $k \geqq 0$ we denote by $Y_{k+1}(\,)$ the function defined by

$$Y_{k+1}(t) = \frac{t^k}{k!} \qquad \text{(for } t \in \mathbf{R}\text{).} \tag{3.44}$$

It is not hard to prove that

$$Y_{k+1} = \frac{D}{D^{k+1}} \qquad \text{(see 4.5) in [7]).} \tag{3.45}$$

3.46. **Lemma.** *If* $G \in \mathfrak{B}$ *and* $n \geqq 1$ *then* $\partial^n([Y_n]^0 \otimes G) = G$.

PROOF. Since $Y_1(\,) = 1(\,)$, we can use 2.31 to obtain

$$\partial([Y_1]^0 \otimes G) = G. \tag{1}$$

Next, observe that the equations

$$[1]^0 \otimes ([Y_n]^0 \otimes G) = ([1]^0 \otimes [Y_n]) \otimes G = [1 \wedge Y_n] \otimes G$$

come from 2.28 and 2.32; since $1 \wedge Y_n(\,) = Y_{n+1}(\,)$ (see 2.33 and 3.44), we have

$$[1]^0 \otimes \big[[Y_n]^0 \otimes G\big] = [Y_{n+1}]^0 \otimes G. \tag{2}$$

The equations

$$\partial^{n+1}([Y_{n+1}]^0 \otimes G) = \partial^{n+1}\big((1)^0 \otimes \big[[Y_n]^0 \otimes S\big]\big) = \partial^n \big[[Y_n]^0 \otimes G\big] \tag{3}$$

are from (2) and 2.31 (with $S = [Y_n]^0 \otimes G$). The property $\partial^k([Y_k]^0 \otimes G) = G$ holds for $k = 1$ (by (1)); it holds for $k = n + 1$ whenever it holds for $k = n$ (by (3)): therefore, it holds for every integer $k \geqq 1$.

§ 4. Initial values of distributions

The following notation and terminology was introduced in [6]. 4.1.

Suppose that $a < 0$ and $A \in \mathscr{A}$ (see 3.3). We say that *A agrees with B on the interval* $(a, 0)$ if $B \in \mathscr{A}$ and

$$.Aw(t) = .Bw(t) \qquad \text{(for } a < t < 0 \text{ and any } w(\,) \text{ in } W\text{).}$$

The relation $A \subset B$ means that there exists some number $a < 0$ such that A agrees with B on the interval $(a, 0)$.

4.2. *The space* $\mathscr{B}_\omega$. As in [7], we denote by $\mathscr{B}_\omega$ the family of all the elements B of $\mathscr{A}$ such that the relation $f \subset B$ holds for some $f(\,)$ in $\mathscr{F}$. Recall that f is the operator of the function $f(\,)$ (see 3.6).

4.3. *Remark*. In consequence of 4.1—4.2, we can say that $B \in \mathscr{B}_\omega$ if (and only if) $B \in \mathscr{A}$ and there exists a function $f(\,)$ in $\mathscr{F}$ and a number $a<0$ such that

$$.Bw(t) = .fw(t) \qquad (\text{for } a < t < 0 \text{ and } w(\,) \in W).$$

4.4. *Derivable operators*. An operator B is said to be *derivable* if $B \in \mathscr{A}$ and if the relation $f \subset B$ holds for some $f(\,)$ in $\mathscr{F}$ such that $|f(0_-)| < \infty$. If B is derivable, there exists a unique complex number $\langle B, 0_- \rangle$ such that the equation $\langle B, 0_- \rangle = f(0_-)$ holds for some function $f(\,)$ in $\mathscr{F}$ such that $f \subset B$ (see [6, 5.0]). We set

$$\partial_t B = BD - \langle B, 0_- \rangle D. \tag{4.5}$$

We define $\partial_t^n B$ recursively by the equations $\partial_t^0 B = B$ and $\partial_t^{k+1} B = \partial_t(\partial_t^k B)$.

4.6. *Remark*. If $f(\,) \in \mathscr{F}$ and if $|f(0_-)| < \infty$ then $\langle f, 0_- \rangle = f(0_-)$: see 2.17 in [7].

4.7. *Reorientation*. One of our aims is to prove that the transformation $R \to R^1$ is an injection of $\mathfrak{B}$ into $\mathscr{B}_\omega$. Our main result will be obtained by relating distributional derivation $R \mapsto \partial R$ to the operation $B \mapsto \partial_t B$ (on operators): as we shall see, if R is an arbitrary distribution such that $\partial R \in \mathfrak{B}$, then $R \in \mathfrak{B}$, the operator R^1 is derivable, and $(\partial R)^1 = \partial_t R^1$; moreover, it is natural to consider $\langle R^1, 0_- \rangle$ as the *initial value* (denoted $R(0_-)$ in the Introduction) of the distribution R.

4.8. **Lemma.** *If* $X \in \mathscr{B}_\omega$ *then* X/D *is derivable, and* $\left\langle \dfrac{X}{D}, 0_- \right\rangle = 0$.

PROOF. Set $G=1$ in [7, 3.5].

4.9. **Theorem.** *If* $F \subset \mathfrak{B}$ *then* $F^1 \in \mathscr{B}_\omega$.

PROOF. From 1.20 and 1.18 it follows the existence of a function $f(\,)$ in $\mathscr{F}$ and a distribution L in $(\mathscr{L})$ such that

$$F^1 = [f]^{01} + L^1 + F_+^1. \tag{1}$$

We intend to prove that $f \subset F^1$. Take any $w(\,)$ in W. From (1) and 3.36 it follows that

$$.F^1 w(\,) = .fw(\,) + (.L^1 w(\,) + .F_1^+ w(\,)). \tag{2}$$

On the other hand, 3.32 gives

$$[.L^1 w + .F_+^1 w]^0 = L \otimes [w']^0 + F_+ \otimes [w']^0. \tag{3}$$

Since $L \in (\mathscr{L})$ it follows from 2.24 that $L \otimes [w']^0$ belongs to $(\mathscr{L})$, so that 1.16 therefore insures the existence of a number $a<0$ such that

$$\mathbf{O}\,(L \otimes [w']^0) \supset (a, \infty). \tag{4}$$

Next, observe that $F_+ \otimes [w']^0$ belongs to $\mathfrak{B}_+$ (by 2.23), so that

$$\mathbf{O}\,(F_+ \otimes [w']^0) \supset (-\infty, 0) \qquad \text{(by 1.13)}. \tag{5}$$

Combining (3), (4), and (5), we can use 1.6 to obtain

$$\mathbf{O}\,([.L_1 w + .F^1_+ w]^0) \supset (a, \infty) \cap (-\infty, 0) = (a, 0).$$

Thus, $[.L^1 w + .F^1_+ w]^0$ equals $\mathbf{0}$ in $(a, 0)$, whence $.L^1 w(\,) + F^1_+ w(\,) = 0$ on the interval $(a, 0)$ (by 1.11); since the functions are continuous we therefore have

$$.L^1 w(t) + .F^1_+ w(t) = 0 \qquad \text{(for } a < t < 0\text{);}$$

consequently, (2) gives

$$(6) \qquad .F^1 w(t) = .fw(t) \qquad \text{(for } a < t < 0\text{).}$$

Since $w(\,)$ is an arbitrary element of W, Equation (6) states that the operator f agrees with F^1 on the interval $(a, 0)$: therefore, $f \subset F^1$. The conclusion $F^1 \in \mathscr{B}_\omega$ is now immediate from Definition 4.2.

4.10. **Theorem.** *If R is a distribution such that ∂R belongs to $\mathfrak{B}$, then R also belongs to $\mathfrak{B}$, the operator R^1 is derivable, and $\partial_t R^1 = (\partial R)^1$.*

PROOF. In view of our hypothesis $\partial R \in \mathfrak{B}$, we can set $S = \partial R$ in 2.31 to obtain

$$\partial([1]^0 \otimes \partial R) = \partial R:$$

thus, both $[1]^0 \otimes \partial R$ and R have the same derivative: therefore, they differ by a constant function [4, p. 328]; thus, there exists a number c such that

$$(1) \qquad R = [f_c]^0 + [1]^0 \otimes \partial R,$$

where $f_c(\,)$ is the constant function $f_c(\,) = c$. Since ∂R belongs to $\mathfrak{B}$ (by hypothesis), it follows from (1) and 2.20 that $R \in \mathfrak{B}$ (recall 1.27). From (1) and 3.39 it results that

$$(2) \qquad R^1 = [f_c]^{01} + \frac{(\partial R)^1}{D} = f_c + \frac{(\partial R)^1}{D}:$$

the second equation is from 3.36. Since f_c is the operator of the constant function $f_c(\,) = c$, it follows from 4.6 that

$$(3) \qquad \langle f_c, 0_- \rangle = c.$$

Note that $(\partial R)^1$ belongs to $\mathscr{B}_\omega$ (by 4.9 and our hypothesis $\partial R \in \mathfrak{B}$); we may therefore set $X = (\partial R)^1$ in 4.8 to obtain

$$(4) \qquad \left\langle \frac{(\partial R)^1}{D}, 0_- \right\rangle = 0.$$

From (2), (3), and (4) we see that

$$\langle R^1, 0_- \rangle = \langle f_c, 0_- \rangle + \left\langle \frac{(\partial R)^1}{D}, 0_- \right\rangle = c + 0 = c.$$

Therefore, 4.5 gives

$$\partial_t R^1 = DR^1 - cD = D\left(f_c + \frac{(\partial R)^1}{D}\right) - f_c D = (\partial R)^1:$$

the middle equation is from (2). Thus, the conclusion $\partial_t R^1 = (\partial R)^1$ is at hand.

4.11. **Lemma.** *Let F be a distribution. The implication*

(5) $$\partial^\nu F \in \mathfrak{B} \quad \textit{implies} \quad \partial_t^\nu F^1 = (\partial^\nu F)^1$$

holds for every integer $\nu \geqq 0$.

Proof. Clearly, (5) holds for $\nu=0$. To proceed by induction, assume that (5) holds for $\nu=k-1$:

(6) $$\partial^{k-1} F \in \mathfrak{B} \quad \text{implies} \quad \partial_t^{k-1} F^1 = (\partial^{\nu-1} F)^1.$$

Let us prove that (5) holds for $\nu=k$. If $\partial^k F \in \mathfrak{B}$ then $\partial(\partial^{k-1} F) \in \mathfrak{B}$, so that 4.10 gives

(7) $$\partial^{k-1} F \quad \text{belongs to} \quad \mathfrak{B}$$

and

(8) $$\partial_t (\partial^{k-1} F)^1 = [\partial (\partial^{k-1} F)]^1.$$

From (7) and (6) we see that

(9) $$\partial_t (\partial_t^{k-1} F^1) = \partial_t (\partial^{k-1} F)^1:$$

combining (9) with (8), we obtain $\partial_t^k F^1 = [\partial(\partial^{k-1} F)]^1$. Thus, (5) holds for $\nu=k$ when it holds for $\nu=k-1$.

4.12. **Lemma.** *Let F be a distribution. If m is an integer* $\geqq 1$ *such that* $\partial^m F \in \mathfrak{B}$, *then*

(4.13) $$\partial^\nu F \in \mathfrak{B} \quad \text{and} \quad \partial_t^\nu F^1 = (\partial^\nu F)^1 \quad \text{for } 0 \leqq \nu \leqq m.$$

Proof. First, suppose that $0 \leqq \nu \leqq m-1$. Let us prove that

(10) $$\partial^{\nu+1} F \in \mathfrak{B} \quad \text{implies} \quad \partial^\nu F \in \mathfrak{B}.$$

To that effect, observe that the hypothesis $(\partial^{\nu+1} F \in \mathfrak{B})$ gives $\partial(\partial^\nu F) \in \mathfrak{B}$, whence the conclusion $\partial^\nu F \in \mathfrak{B}$ follows immediately

4.10. Consequently, we have proved that

(11) $$0 \leqq \nu \leqq m-1 \quad \text{and} \quad \partial^\nu F \notin \mathfrak{B} \quad \text{implies} \quad \partial^{\nu+1} F \notin \mathfrak{B}.$$

Next, let M be the set of all the non-negative integers $\nu \leqq m$ such that $\partial^\nu F \notin \mathfrak{B}$: since $\partial^m F \in \mathfrak{B}$ (by hypothesis), we have $m \notin M$, whence

(12) $$\nu \in M \quad \text{implies} \quad 0 \leqq \nu \leqq m-1.$$

In view of 4.11, our conclusion 4.13 can be obtained by proving that

(13) $$0 \leqq \nu \leqq m \quad \text{implies} \quad \partial^\nu F \in \mathfrak{B}.$$

We shall prove (13) by contradiction: if (13) fails, then M is a non-void subset of the positive integers: it therefore follows (from (12)) the existence of an integer

(14) $$\nu = \max M \qquad \text{(see [3, p. 64])}.$$

Since $\nu \in M$, it follows from (12) that $0 \leqq \nu \leqq m-1$. Since $\nu \in M$, we have $\partial^\nu F \notin \mathfrak{B}$: from (11) it now follows that $\partial^{\nu+1} F \notin \mathfrak{B}$. Consequently, $0 \leqq \nu+1 \leqq m$ and $\partial^{\nu+1} F \notin \mathfrak{B}$: therefore, $\nu+1$ belongs to M, which contradicts (14). This contradiction establishes (13); from (13) and 4.11 now follows 4.13.

§ 5. The main result

Throughout this section, μ is a polynomial μ_k $(k=0, 1, 2, \dots)$ of degree $m \geqq 1$ (that is, μ is a sequence of complex numbers such that $\mu_m \neq 0$ and $\mu_k = 0$ for $k > m$). As usual,

(5.1) $$\mu(D) = \mu_0 + \mu_1 D + \dots + \mu_m D^m.$$

If u is a distribution we set

(5.2) $$\mu(\partial)u = \mu_0 u + \mu_1 \partial u + \dots + \mu_m \partial^m u.$$

If λ is a polynomial whose degree is smaller than m, we denote by $g^\lambda()$ the unique continuous function $h_m()$ such that $D\lambda(D)/\mu(D) = h_m$ (see [7, 5.4]):

(5.3) $$D\frac{\lambda(D)}{\mu(D)} = g_\mu^\lambda.$$

5.4 *Remark*. In the particular case where λ is the polynomial such that $\lambda(D)=1$, we set $g_u = g_\mu^\lambda$, so that

(5.5) $$\frac{D}{\mu(D)} = g_\mu.$$

If $S \in \mathfrak{B}$ then

(5.6) $$\frac{S^1}{\mu(D)} = [[g_\mu]^0 \otimes S]^1.$$

To prove 5.6 it suffices to note the equations

$$[[g_\mu]^0 \otimes S]^1 = g_\mu D^{-1} S^1 = \frac{D}{\mu(D)} D^{-1} S^1 = \frac{S^1}{\mu(D)}$$

come from 3.38 and 5.5.

5.7. **Lemma.** *If* $S \in \mathfrak{B}$ *the equation*

(5.8) $$F = [g_\mu]^0 \otimes S$$

determines a solution of the initial-value problem

(5.9) $$\langle \partial_t^k F^1, 0_- \rangle = 0 \qquad (\text{for } 0 \leqq k \leqq m-1)$$

and

(5.10) $$\mu(\partial) F = S;$$

moreover,

(5.11) $$F \in \mathfrak{B} \quad \text{and} \quad \partial^\nu F \in \mathfrak{B} \quad \text{for all } \nu \leqq m.$$

Proof. From 5.8 and 5.6 we see that

(1) $$F^1 = \frac{S^1}{\mu(D)},$$

so that 5.9 is obtained by setting $B=S^1$ in [7, (5.7)] (the property $B\in\mathscr{B}_\omega$ comes from 4.9). It only remains to prove 5.10—5.11. From (1) and [7, (5.8)] (again with $B=S^1$) we also obtain

$$\mu(\partial_t)F^1 = S^1. \tag{2}$$

Next, the equations

$$F^1 = \frac{D^m}{\mu(D)}\frac{S^1}{D^m} = \left(\frac{1}{\mu_m}+g_m D^{-1}\right)\frac{S^1}{D_m}$$

are from (1) and [7, (5.5)]; therefore,

$$F^1 = \frac{D}{D^m}D^{-1}\left(\frac{1}{\mu_m}S^1+g_m D^{-1}S^1\right) \tag{3}$$

now set

$$G = \frac{1}{\mu_m}S+[g_m]^0\otimes S. \tag{4}$$

From (4) and 3.38 we obtain $G^1=\mu_m^{-1}S^1+g_m D^{-1}S^1$: substituting into (3), we can use 3.45 to write

$$F^1 = Y_m D^{-1}G^1 = ([Y_m]^0\otimes G)^1: \tag{5}$$

the last equation is from 3.38. Since $g_m(\,)\in\mathscr{F}$ we have $[g_m]^0\in\mathfrak{B}$ (by 1.27), so that $[g_m]^0\otimes S$ belongs also to $\mathfrak{B}$ (by 2.20); since $S\in\mathfrak{B}$ it now follows directly from (4) that

$$G \text{ belongs to } \mathfrak{B}. \tag{6}$$

In view of (6), we see that $[Y_m]^0\otimes G$ belongs to $\mathfrak{B}$ (by 2.20); since $F\in\mathfrak{B}$ (by 5.8 and 2.20), we can therefore apply 3.42 to (5):

$$F = [Y_m]^0\otimes G \quad \text{(from (5))}. \tag{7}$$

From (7) and 3.46 it now follows that $\partial^m F=G$; since $G\in\mathfrak{B}$ it results from 4.12 that

$$\partial^\nu F\in\mathfrak{B} \quad\text{and}\quad (\partial^\nu F)^1 = \partial_t^\nu F^1 \quad (\text{or } 0\leqq\nu\leqq m). \tag{8}$$

From (8) and 5.2 we see that

$$\big(\mu(\partial)F\big)^1 = \mu(\partial_t)F^1 = S^1: \tag{9}$$

the second equation is from (2). From (8) it also follows that $\mu(\partial)F$ belongs to $\mathfrak{B}$; we may therefore use 3.42 to infer from (9) that $\mu(\partial)F=S$.

This establishes 5.10. Conclusion 5.11 is immediate from (8), and 5.9 was verified at the beginning of this proof.

5.12. **First main theorem.** *Suppose that $S\in\mathfrak{B}$. If u is a distribution such that $\mu(\partial)u=S$, then $u\in\mathfrak{B}$,*

$$\partial^k u\in\mathfrak{B}, \quad\text{and}\quad (\partial^k u)^1 = \partial_t^k u^1 \quad (\text{for } 0\leqq k\leqq m); \tag{5.13}$$

moreover,

$$(\partial^k u)^1 = D^k u^1-\sum_{\nu=1}^{k-1}\langle\partial_t^\nu u^1, 0_-\rangle D^{k-\nu} \quad (\text{for } 1\leqq k\leqq m). \tag{5.14}$$

PROOF. Let F be as in 5.8. Since $\mu(\partial)F=S$ and $\mu(\partial)u=S$ it follows from [4, p. 328] the existence of a polynomial $p(\,)$ of degree $\leqq m-1$ such that $u=F+[p]^0$: consequently, $u\in\mathfrak{B}$ and

$$\partial^m u = \partial^m F+\partial^m[p]^0 = \partial^m F:$$

the last equation is from the fact that $\partial^m[p]^0=p^{(m)}=\mathbf{0}$ (since the degree of $p(\,)$ is $\leqq m-1$). Thus, $\partial^m u=\partial^m F$; but $\partial^m F\in\mathfrak{B}$ (by 5.11), so that $\partial^m u\in\mathfrak{B}$: Conclusion 5.13 is now immediate from 4.12. Conclusion 5.14 comes from 5.13 and [7, 4.1].

5.15. **Second main theorem.** *Let c_ν $(\nu=0, 1, \ldots, m-1)$ be given numbers. If $S\in\mathfrak{B}$ there exists a unique distribution u such that*

(5.16) $$\mu(\partial)u = S$$

and

(5.17) $$\langle\partial_t^\nu u^1, 0_-\rangle = c_\nu \quad (\text{for } 0\leqq\nu<m).$$

That solution u belongs to $\mathfrak{B}$, satisfies 5.13, and is determined by the equation

(5.18) $$u = [g_\mu]^0\otimes S+[g_\mu^\lambda]^0,$$

where $g_\lambda^\mu(\,)$ is the function determined by

(5.19) $$g_\mu^\lambda = \frac{1}{\mu(D)}\sum_{n=1}^{m}\mu_n\sum_{\nu=0}^{n-1}c_\nu D^{n-\nu}.$$

PROOF. First, we verify the existence of a solution of the initial-values problem 5.16—5.17. Let $y(\,)$ be the solution of the initial-value problem

(1) $$\sum_{k=0}^{m}\mu_k y^{(k)}(t) = 0 \quad (\text{for } t\in\mathbf{R})$$

subject to the initial conditions

(2) $$y^{(k)}(0) = c_k \quad (\text{for } 0\leqq k<m).$$

It is well-known that such a function $y(\,)$ exists; in fact, $y(\,)$ is infinitely differentiable; consequently, (2) and [7, 2.21] give

(3) $$\langle\partial_t^k y, 0_-\rangle = c_k \quad (\text{for } 0\leqq k<m).$$

Since $y(\,)$ is infinitely differentiable, (1) implies that

(4) $$\mu(\partial)[y]^0 = 0.$$

Let F be the distribution defined by 5.8; we denote by u the distribution

(5) $$u = F+[y]^0.$$

The equations

(6) $$\mu(\partial)u = \mu(\partial)F+\mu(\partial)[y]^0 = S+0 = S$$

are from (5), 5.10, and (4). On the other hand, (5) and 3.36 give $u^1 = F^1 + y$: consequently,

(7) $$\langle \partial_t^k u^1, 0_- \rangle = \langle \partial_t^k F^1, 0_- \rangle + \langle \partial_t^k y, 0_- \rangle = 0 + c_k = c_k:$$

the last two equations are from 5.9 and (3). From (6)—(7) it follows that (5) defines a distribution u satisfying 5.16—5.17.

Finally, we verify the uniqueness. If u is a distribution satisfying 5.16—5.17, then (since $S \in \mathfrak{B}$) we can use 5.14 to obtain

$$(\mu(\partial)u)^1 = \mu(G)u^1 - \sum_{k=1}^{m} \mu_k \sum_{\nu=0}^{k-1} \langle \partial^\nu u^1, 0_- \rangle D^{k-\nu};$$

therefore, it results from 5.16—5.17 that

$$S^1 = \mu(D)u^1 - \sum_{k=1}^{m} \mu_k \sum_{\nu=0}^{k-1} c_\nu D^{k-\nu}:$$

solving for u^1, we can use 5.19 to write

(8) $$u_1 = \frac{S^1}{\mu(D)} + g_\mu^\lambda = ([g_\mu]^0 \otimes S + [g_\mu^\lambda]^0)^1:$$

the second equation is from 5.6 and 3.36. Thus, if u is a distribution satisfying 5.16—5.17, it follows from (8) and 3.42 that u is given by 5.18. From 5.12 it results that $u \in \mathfrak{B}$ and u satisfies 5.13. Since we have already verified the existence of a distribution u satisfying 5.16—5.17, our proof is complete.

§ 6. Particular cases

If $b \geqq 0$ we denote by $T_b(\,)$ the function defined by

$$T_b(t) = \begin{cases} 0 & \text{for } t < b \\ 1 & \text{for } t \geqq b \end{cases}.$$

If $a < 0$ we set

$$T_a(t) = \begin{cases} -1 & \text{for } t < a \\ 0 & \text{for } t \geqq a \end{cases}$$

As usual, δ_x is the distribution defined by $\delta_x(\varphi) = \varphi(x)$ (for each $\varphi(\,)$ in D). We set

(6.1) $$E = \sum_{k=-\infty}^{\infty} \delta_{2k\pi}:$$

this series converges in the topology of D'.

6.2. **Lemma.** *E belongs to $\mathfrak{B}$ and*

(6.3) $$E^1 = D \sum_{k=-\infty}^{\infty} T_{2k\pi}.$$

PROOF. Note that $E=A+B$, where

$$A = \sum_{k=-\infty}^{-1} \delta_{2k\pi} \quad \text{and} \quad B = \sum_{k=0}^{\infty} \delta_{2k\pi}:$$

It is easily verified that $A\in(\mathscr{L})$ and $B\in\mathfrak{B}_+$, so that $E\in\mathfrak{B}$. Next, observe that $\delta_x=\partial[T_x]^0$ for any x in $\mathbf{R}$; therefore, 6.1 gives

(1) $$E = \sum_{k=-\infty}^{\infty} \partial[T_{2k\pi}]^0 = \partial[f]^0,$$

where

(2) $$f(\,) = \sum_{k=-\infty}^{\infty} T_{2k\pi}(\,) \quad \text{(see [8, p. 37])}.$$

Since $E\in\mathfrak{B}$, the equations $E^1=\partial_t[f]^{01}=\partial_t f$ follow from (1), 4.10, and 3.36; but $\partial_t f=Df$ (by [6, 5.8]): therefore, $E^1=Df$. Conclusion 6.3 now comes from [6, 4.12 (with $g=1$)].

6.4. *First example.* To find a distribution u such that

(3) $$\partial^2 u+u = E.$$

Since $E\in\mathfrak{B}$ it follows from 5.12 that $u\in\mathfrak{B}$ and $(\partial^2 u)^1=\partial_t^2 u^1$; consequently, (3) gives

(4) $$\partial_t^2 u^1+u^1 = E^1 = D\sum_{k=-\infty}^{\infty} T_{2k\pi}:$$

the second equation is from 6.3. The equation (4) is precisely the one that has been solved in [6, 6.7]: the explicit solution is given in the introduction of the present paper.

6.5. *Second example.* In case $S=\mathbf{0}$ and $c_{m-1}=1/\mu_m$ with $c_k=0$ for $0\leqq k\leqq m-2$' it follows from 5.18—5.19 that the solution of the problem 5.16—5.17 is given by $u=[g_\mu]^0$, where $g_\mu(\,)$ is the function defined by 5.5: it is the Green's function of the problem.

6.6. *Third example.* In case $S=\delta$ and $c_k=0$ for $0\leqq k\leqq m-1$, it follows from 5.18 that the solution of the problem 5.16—5.17 is given by

$$u = [g_\mu]^0\otimes\delta = [g_\mu]^0_+:$$

the second equation is from 2.29; equivalently,

$$u(t) = \begin{cases} 0 & \text{for } t<0 \\ g_\mu(t) & \text{for } t>0: \end{cases}$$

see 1.28 and 1.26.

6.7. *Fourth example.* In case $B\subset\mathfrak{B}_+$ then $B=\mathfrak{B}_+$ (see 1.21); it follows from 2.22 that the equation 5.18 becomes

$$u = [g_\mu]^0_+ * B+[g_\mu^\lambda]^0.$$

Added in proof. The results in this paper have been generalized by H. Shultz, An algebra of distriburions on an open interval. *Transactions of the American Math. Soc.* **169** (1972), 163—181.

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