Direct decompositions of lattices, rings, and semigroups of continuous functions

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1. Introduction

If X is a topological space, then $C(X)$ will denote, variously, the lattice, ring, or (multiplicative) semigroup of all real-valued continuous functions on X.

If X is the topological sum of a family $(X_\alpha)_{\alpha\in I}$ of subsets of X (or, more generally, if $(X_\alpha)_{\alpha\in I}$ is a pairwise disjoint family of open-and-closed subsets of X such that $\bigcup_{\alpha\in I} X_\alpha$ is strictly C-embedded in X), then it is easily verified that $C(X)$ is lattice, ring, and semigroup isomorphic to the (unrestricted) direct product $\prod_{\alpha\in I} C(X_\alpha)$. (A subset S of X is *C-embedded* (resp. *strictly C-embedded*) in X in case every $f\in C(S)$ has an extension (resp. unique extension) in $C(X)$.) In this note we prove the following converse: If $C(X)$ is decomposable as a product $\prod_{\alpha\in I} A_\alpha$ (of either lattices, rings, or semigroups), then there is a pairwise disjoint family $(X_\alpha)_{\alpha\in I}$ of open-and-closed subsets of X such that (a) each A_α is isomorphic to $C(X_\alpha)$ and (b) $\bigcup_{\alpha\in I} X_\alpha$ is strictly C-embedded in X; and if the cardinal of I is nonmeasurable, then $X=\bigcup_{\alpha\in I} X_\alpha$ (so X is the topological sum of $(X_\alpha)_{\alpha\in I}$) (see 3.1 below). This result, for the special case in which $C(X)$ is the product of *two* lattices, was proved by the author and C. W. BURRILL in [2], Theorem B, and elegantly reproved by S. D. SHORE in [7]. (In turn, Theorem B of [2] is a generalization of an earlier theorem of KAPLANSKY [5], Theorem 2.) Our result, for rings, includes a theorem of S. WARNER [8], p. 78. (Warner's proof relies on a general theorem about characters of direct products of algebras over an infinite field, due independently to Warner [8], Theorem B and to BIAŁYNICKI—BIRULA and ŻELAZKO [1]. Our proof, obtained independently before the publication of [8], is self-contained within the theory of rings of continuous functions; for the latter, see [3].) The result for semigroups is an easy consequence of that for rings.

If X is completely regular, then (b) above can be replaced by the more familiar condition (b′): $\bigcup_{\alpha\in I} X_\alpha$ is dense and C-embedded in X. If complete regularity is omitted, however, and if there is a measurable cardinal, then (b) cannot be replaced by (b′) (see 4.2). (We are indebted to A. W. HAGER for suggesting that explicit consideration be given to the case in which the cardinal $|I|$ is measurable, and for proposing the precise formulation of (b').)

In § 5 we show that our result leads to an equivalent formulation of Ulam's axiom (i.e., the assertion, known to be consistent with the usual axioms of set theory, that every cardinal is nonmeasurable).

Except where specified, no separation properties will be required of X.

2. Preliminaries

We first summarize the definitions and results that will be needed from [3]. An ideal M of the ring $C(X)$ is *real* in case the quotient ring $C(X)/M$ is isomorphic to the field $\mathbf{R}$ of real numbers. If $p\in X$, then clearly

$$M_p = \{f\in C(X)\colon f(p) = 0\}$$

is a real ideal of $C(X)$. A completely regular Hausdorff space X is *realcompact* in case each real ideal of $C(X)$ is of the form M_p for some (necessarily unique) $p\in X$. (For equivalent formulations of realcompactness, see [3].)

A cardinal $\mathfrak{m}$ is *measurable* in case there exists a countably additive $\{0, 1\}$-valued measure μ defined on the set of all subsets of a set S of cardinality $\mathfrak{m}$ such that $\mu(S)=1$ and $\mu(\{x\})=0$ for every $x\in S$. (If there is a measurable cardinal, then it is known that the smallest such is strongly inaccessible.) For extensive discussions concerning measurable and nonmeasurable cardinals, see [3; Chap. 12] and [6].

2.1. Proposition.

(a) [3; 8.10]. A closed subspace of a realcompact space is itself realcompact.

(b) [3; 12G]. If X is the topological sum of a family $(X_\alpha)_{\alpha\in I}$ of realcompact subspaces, and if $|I|$ is nonmeasurable, then X is realcompact.

(c) [3; 12.2]. Every realcompact discrete space is of nonmeasurable power.

2.2. Proposition [3; 8.7]. If X is completely regular Hausdorff, then there exists a realcompact space υX (the "Hewitt realcompactification" of X) such that X is ednse and (strictly) C-embedded in υX.

If $Y\subset X$, then the *restriction homomorphism* $\varrho\colon C(X)\to C(Y)$ is defined by $\varrho(f)=f|Y\,(f\in C(X))$.

2.3. Proposition. Let X be a realcompact space, let $(X_\alpha)_{\alpha\in I}$ be a pairwise disjoint family of closed subsets of X, and let $Y=\bigcup_{\alpha\in I} X_\alpha$. If each X_α is open in Y, if $|I|$ is nonmeasurable, and if the restriction homomorphism $\varrho\colon C(X)\to C(Y)$ is bijective, then $X=Y$.

Proof. By 2.1 (a) and (b), Y is realcompact. If $p\in X$, then (since ϱ is an isomorphism onto $C(Y)$) $\varrho(M_p)$ is a real ideal of $C(Y)$. But then $\varrho(M_p)=\{f\in C(Y)\colon f(q)=0\}$ for some $q\in Y$, and obviously $p=q$. Thus $X=Y$.

3. Decompositions of $C(X)$ as a lattice, ring, and semigroup

We can now state the main result:

3.1. **Theorem.** *Let X be a topological space, let $(A_\alpha)_{\alpha\in I}$ be a family of lattices (resp. rings, resp. semigroups), and assume there is an isomorphism φ from the direct product $A=\prod_{\alpha\in I} A_\alpha$ onto the lattice (resp. ring, resp. multiplicative semigroup) $C(X)$. Then there is a pairwise disjoint family $(X_\alpha)_{\alpha\in I}$ of open-and-closed subsets of X such that* (a) *each A_α is isomorphic to $C(X_\alpha)$ (by an isomorphism described explicitly below) and* (b) *$\bigcup_{\alpha\in I} X_\alpha$ is strictly C-embedded in X. Moreover, if $|I|$ is nonmeasurable, then $X=\bigcup_{\alpha\in I} X_\alpha$ (so X is the topological sum of $(X_\alpha)_{\alpha\in I}$).*

Proof. We first give the proof for lattices: For each $\alpha\in I$, let $A_\alpha^*=\prod_{\beta\neq\alpha} A_\beta$ and let φ_α be the canonical isomorphism from $A_\alpha\times A_\alpha^*$ onto A. For each $x\in X$, set

$$P_x=\{f\in C(X)\colon f(x)\leqq 0\};$$

and for each $\alpha\in I$, denote by X_α the set of all $x\in X$ with the property that $(\varphi\circ\varphi_\alpha)^{-1}(P_x)$ is of the form $P_\alpha\times A_\alpha^*$ for some prime ideal P_α of the lattice A_α. If $\alpha\neq\beta$, then (as one easily verifies) X_α and X_β are disjoint.

If we appeal now to [2] (especially Theorem B, and its proof, and Remarks (1) and (2) of [2]), the isomorphism

$$\varphi\circ\varphi_\alpha\colon\ A_\alpha\times A_\alpha^*\to C(X)$$

yields the following three facts:

(i) X_α is open-and-closed in X.

(ii) If i_α is an (arbitrary) injection of A_α into A such that $\mathrm{pr}_\alpha\circ i_\alpha$ is the identity map of A_α (where pr_α denotes the projection of the product A of index α), and if $\varrho_\alpha\colon C(X)\to C(X_\alpha)$ is the restriction homomorphism, then

$$\varrho_\alpha\circ\varphi\circ i_\alpha\colon\ A_\alpha\to C(X_\alpha)$$

is an isomorphism.

(iii) For every $f\in C(X)$ and every $\alpha\in I$,

$$(*)\qquad (\varrho_\alpha\circ\varphi\circ i_\alpha)\big(\mathrm{pr}_\alpha(\varphi^{-1}(f))\big)=\varrho_\alpha(f).$$

Now let $Y=\bigcup_{\alpha\in I} X_\alpha$ and let $\varrho\colon C(X)\to C(Y)$ be the restriction homomorphism from $C(X)$ into $C(Y)$. Then for each $f\in C(X)$ and each $\alpha\in I$, $\varrho(f)|X_\alpha=\varrho_\alpha(f)$; and from this, (ii), and (*) it follows easily that ϱ is a bijection from $C(X)$ onto $C(Y)$. Thus Y is strictly C-embedded in X.

To complete the proof in the lattice case, assume now that $|I|$ is nonmeasurable. By a result due to Stone and to Čech (see e.g. [3; 3.9]), there exists a completely regular Hausdorff space X' and a continuous surjection $\tau\colon X\to X'$ such that the map defined by $g\to g\circ\tau$ is a ring (and hence also lattice and semigroup) isomorphism from $C(X')$ onto $C(X)$. Moreover (see 2.2), the map given by $g\to g|X'$ is an isomorphism from $C(\upsilon X')$ onto $C(X')$. Combining these facts, one readily verifies that (for the purpose of proving the final assertion of the theorem) we may assume that X is realcompact. But then $X=Y$ by 2.3, so the proof in this case is complete.

We turn next to the ring case. (While this case could doubtless be reduced to the former, it seems simpler, and more instructive, to give a direct argument.) Let $i_\alpha: A_\alpha \to A$ be the canonical injection (i.e., $\mathrm{pr}_\alpha \circ i_\alpha$ is the identity map of A_α and $\mathrm{pr}_\beta \circ i_\alpha = 0$ for all $\beta \neq \alpha$), and note that each A_α obviously has a unit element 1_α. Then, for each $\alpha \in I$,

$$h_\alpha = (\varphi \circ i_\alpha)(1_\alpha)$$

is an idempotent in the ring $C(X)$ so h_α is the characteristic function of an open-and-closed subset X_α of X. Moreover, if $\alpha \neq \beta$, then $X_\alpha \cap X_\beta = \emptyset$.

Once again, let $\varrho_\alpha: C(X) \to C(X_\alpha)$ be the restriction homomorphism. We show that (*) above holds in the present case:

Case 1. $f=0$ *on* $X-X_\alpha$. For each $\beta \neq \alpha$, we have $f \cdot h_\beta = 0$, and hence

$$\begin{aligned}
\mathrm{pr}_\beta(\varphi^{-1}(f)) &= \mathrm{pr}_\beta(\varphi^{-1}(f)) \cdot \mathrm{pr}_\beta(i_\beta(1_\beta)) \\
&= \mathrm{pr}_\beta(\varphi^{-1}(f) \cdot i_\beta(1_\beta)) \\
&= \mathrm{pr}_\beta(\varphi^{-1}(f) \cdot \varphi^{-1}(h_\beta)) \\
&= (\mathrm{pr}_\beta \circ \varphi^{-1})(f \cdot h_\beta) = 0.
\end{aligned}$$

It follows that $(i_\alpha \circ \mathrm{pr}_\alpha)(\varphi^{-1}(f)) = \varphi^{-1}(f)$, from which (*) is clear.

Case 2. f *arbitrary in* $C(X)$. Apply Case 1 (to the functions h_α and $f \cdot h_\alpha$) to obtain

$$\begin{aligned}
(\varrho_\alpha \circ \varphi \circ i_\alpha)(\mathrm{pr}_\alpha(\varphi^{-1}(f))) &= (\varrho_\alpha \circ \varphi \circ i_\alpha)(\mathrm{pr}_\alpha(\varphi^{-1}(f))) \cdot \varrho_\alpha(h_\alpha) \\
&= (\varrho_\alpha \circ \varphi \circ i_\alpha)(\mathrm{pr}_\alpha(\varphi^{-1}(f \cdot h_\alpha))) \\
&= \varrho_\alpha(f \cdot h_\alpha) = \varrho_\alpha(f),
\end{aligned}$$

and we have proved (*).

Now it is a routine matter to verify that the map

$$\varrho_\alpha \circ \varphi \circ i_\alpha : A_\alpha \to C(X_\alpha)$$

is injective. Moreover, by (*) and the fact that X_α is (trivially) C-embedded in X, this map is surjective, and hence an isomorphism. Thus (i), (ii), and (iii) hold again, so the proof can be completed exactly as before.

Finally, the semigroup case is easily reduced to the ring case: It will suffice to show that addition can be defined on each multiplicative semigroup A_α in such a way that A_α becomes a ring and φ becomes a ring isomorphism.

Note first that each A_α necessarily has a zero-element and a unit element 1_α. Define $i_\alpha: A_\alpha \to A$ as in the ring case, and define addition on (the underlying set of) A_α as follows: For $a_\alpha, b_\alpha \in A_\alpha$,

$$a_\alpha + b_\alpha = (\mathrm{pr}_\alpha \circ \varphi^{-1})(\varphi(i_\alpha(a_\alpha)) + \varphi(i_\alpha(b_\alpha))).$$

Addition in A is then defined coordinatewise. It will now suffice to show that, for every $a, b\in A$, $\varphi(a+b)=\varphi(a)+\varphi(b)$; and this is equivalent to the following: for every $a, b\in A$ and every $\alpha\in I$,

$$\mathrm{pr}_\alpha(a)+\mathrm{pr}_\alpha(b) = (\mathrm{pr}_\alpha\circ\varphi^{-1})\big(\varphi(a)+\varphi(b)\big).$$

To verify the latter, we calculate as follows:

$$\begin{aligned}\mathrm{pr}_\alpha(a)+\mathrm{pr}_\alpha(b) &= (\mathrm{pr}_\alpha\circ\varphi^{-1})\big((\varphi\circ i_\alpha)(\mathrm{pr}_\alpha(a))+(\varphi\circ i_\alpha)(\mathrm{pr}_\alpha(b))\big)\\ &= (\mathrm{pr}_\alpha\circ\varphi^{-1})\big(\varphi(a\cdot i_\alpha(1_\alpha))+\varphi(b\cdot i_\alpha(1_\alpha))\big)\\ &= (\mathrm{pr}_\alpha\circ\varphi^{-1})\big(\varphi(a)\cdot(\varphi\circ i_\alpha)(1_\alpha)+\varphi(b)\cdot(\varphi\circ i_\alpha)(1_\alpha)\big)\\ &= (\mathrm{pr}_\alpha\circ\varphi^{-1})\big((\varphi(a)+\varphi(b))\cdot(\varphi\circ i_\alpha)(1_\alpha)\big)\\ &= \mathrm{pr}_\alpha\big(\varphi^{-1}(\varphi(a)+\varphi(b))\cdot i_\alpha(1_\alpha)\big)\\ &= (\mathrm{pr}_\alpha\circ\varphi^{-1})\big(\varphi(a)+\varphi(b)\big).\end{aligned}$$

The proof is now complete.

4. The role of complete regularity

The next proposition shows that if X is completely regular, then (b) of 3.1 can be replaced by (b'): $\bigcup_{\alpha\in I} X_\alpha$ *is dense and C-embedded in X.*

4.1. Proposition. If X is completely regular and $Y\subset X$, then these are equivalent:

(1) Y is strictly C-embedded in X.

(2) Y is dense and C-embedded in X.

Proof. Assume (1), let $x\in X$, and let V be a neighborhood of x. Choose $f\in C(X)$ with $f(x)=1$ and $f=0$ on $X-V$. If $V\cap Y=\emptyset$, then f and the constant function 0 are distinct extensions of the same function on Y. Hence V meets Y, so (2) holds. The converse is clear.

The implication (2)⇒(1) of 4.1 holds, of course, without complete regularity, but in its absence (1)⇒(2) may fail: Let X be a regular T_1-space (with more than one point) such that every $f\in C(X)$ is constant (see [4]); pick any $x\in X$ and let $Y=\{x\}$.

If the cardinal of I in 3.1 is nonmeasurable, then (by the final assertion of 3.1), (b) and (b') trivially coincide. We now show that if there is a measurable cardinal, then, in general, (b') cannot replace (b) in 3.1. (The space we construct for this purpose is merely T_0. It seems likely that a more refined technique would provide a similar example with stronger separation.)

4.2. Example. If $\mathfrak{m}$ is a measurable cardinal, then there is a T_0-space X, a discrete subspace Y of X with $|Y|=\mathfrak{m}$, and a lattice, ring, and semigroup isomorphism φ from $A=\prod_{y\in Y} C(\{y\})$ onto $C(X)$ with this property: If $(X_y)_{y\in Y}$ is any family of subsets of X such that, for every $y\in Y$, $\varrho_y\circ\varphi\circ i_y\colon C(\{y\})\to C(X_y)$ is an isomorphism (where $i_y\colon C(\{y\})\to A$ is the canonical injection and $\varrho_y\colon C(X)\to C(X_y)$ is the restriction homomorphism), then $\bigcup_{y\in Y} X_y$ is not dense in X, but is strictly C-embedded in X.

Proof. Let Y be any discrete space with $|Y|=\mathfrak{m}$. Since Y is not realcompact (2.1 (c)), there is a point $p\in \upsilon Y-Y$. Choose an arbitrary $q\notin \upsilon Y$ and let

$$X = \upsilon Y\cup\{q\}.$$

Topologize the set X as follows: For each $x\in\upsilon Y$, denote by $\mathscr{V}(x)$ the filter of neighborhoods of x in υY. Then, for each $x\in X$, a base for the filter of neighborhoods of x in X is declared to be $\mathscr{B}(x)$, where

$$\mathscr{B}(x) = \begin{cases}\mathscr{V}(x) & \text{if} \quad x\notin\{p,q\},\\ \{V\cup\{q\}\colon\ V\in\mathscr{V}(x)\} & \text{if} \quad x=p,\\ \{\{q\}\} & \text{if} \quad x=q.\end{cases}$$

With this topology, X is evidently a T_0-space such that υY is a subspace of X. Moreover, it is clear that

$$f(p)=f(q) \quad \text{for every} \quad f\in C(X). \tag{1}$$

For each $g\in C(\upsilon Y)$, define $\theta(g)\in\mathbf{R}^X$ as follows: If $x\in X$, then

$$\theta(g)(x) = \begin{cases} g(x) & \text{if} \quad x\neq q,\\ g(p) & \text{if} \quad x=q.\end{cases}$$

It is readily verified that if $g\in C(\upsilon Y)$, then $\theta(g)\in C(X)$ and that the map $\theta\colon C(\upsilon Y)\to$ $\to C(X)$ given by $g\to\theta(g)$ is, in fact, a lattice, ring, and semigroup monomorphism. Denote by $\upsilon\colon C(Y)\to C(\upsilon Y)$ the map that associates with each $h\in C(Y)$ its unique continuous extension $\upsilon(h)$ over υY (see 2.2). If $f\in C(X)$, then clearly $f|\upsilon Y=\upsilon(f|Y)$; and from this and (1) it follows that

$$f=\theta(\upsilon(f|Y)) \quad \text{for every } f\in C(X). \tag{2}$$

Thus θ is an isomorphism from $C(\upsilon Y)$ onto $C(X)$.

Next, since Y is discrete, there is an isomorphism $\psi\colon A\to C(Y)$ such that, for every $a\in A$ and every $y\in Y$,

$$\psi(a)(y) = \mathrm{pr}_y(a)(y). \tag{3}$$

Then $\varphi=\theta\circ\upsilon\circ\psi$ is an isomorphism from A onto $C(X)$.

Suppose now that $(X_y)_{y\in Y}$ is a family of subsets of X such that, for each $y\in Y$, $\varrho_y\circ\varphi\circ i_y\colon C(\{y\})\to C(X_y)$ is an isomorphism (where i_y and ϱ_y are as described in the statement of 4.2). For each $y\in Y$, let 1_y be the unit element $C(\{y\})$ and let χ_y be the characteristic function on X of $\{y\}$. Then, since Y is dense in υY and $p\in\upsilon Y-Y$, one can verify that (3) and (1) imply that

$$\varphi(i_y(1_y))=\chi_y \text{ for every } y\in Y.$$

Now if there is a point $x\in X_y$ such that $x\neq y$, then we have

$$1=(\varrho_y\circ\varphi\circ i_y)(1_y)(x)=\varrho_y(\chi_y)(x)=\chi_y(x)=0,$$

a contradiction. Since X_y is obviously nonempty, it follows that $X_y=\{y\}$, and thus $\bigcup_{y\in Y} X_y$ is precisely the set Y.

To complete the proof, note first that if $h\in C(Y)$, then, by (2),

$$\theta(\upsilon(h)) = \theta(\upsilon(\theta(\upsilon(h))|Y)),$$

and therefore $h=\theta(\upsilon(h))|Y$. Thus every $h\in C(Y)$ has a continuous extension over X, and by (2) this extension is unique. Finally, since q is not in the closure of Y, Y is not dense in X.

5. Equivalent formulations of Ulam's axiom

We show, in conclusion, that 2.3 and (the final assertion of) 3.1 each lead to an equivalent formulation of Ulam's axiom (see § 1). (As a consequence, each of these results can fail if the cardinal of the index set I is measurable.)

5.1. **Theorem.** *These are equivalent:*

(1) *Ulam's axiom.*

(2) *If X is any realcompact space, if $(X_\alpha)_{\alpha\in I}$ is any pairwise disjoint family of closed subsets of X such that each X_α is open in $Y=\bigcup_{\alpha\in I} X_\alpha$, and if the restriction homomorphism $\varrho\colon C(X)\to C(Y)$ is bijective, then $X=Y$.*

(3) *If X is any (realcompact) space, if $(A_\alpha)_{\alpha\in I}$ is any family of lattices (resp. rings, resp. semigroups), and if the lattice (resp. ring, resp. semigroup) $C(X)$ is isomorphic to the direct product $\prod_{\alpha\in I} A_\alpha$, then X is the topological sum of a family $(X_\alpha)_{\alpha\in I}$ of subsets of X with the property that each A_α is isomorphic to the corresponding $C(X_\alpha)$.*

PROOF. The implications (1)⇒(2) and (2)⇒(3) follow from the proofs of 2.3 and 3.1, respectively. To complete the proof, assume (3), let $\mathfrak{m}$ be any cardinal, and let Y be a discrete space with $|Y|=\mathfrak{m}$. By 2.2, $C(\upsilon Y)$ is lattice, ring, and semigroup isomorphic to $C(Y)$. Moreover, $C(Y)$ is isomorphic to $\prod_{y\in Y} C(\{y\})$. By hypothesis, therefore, υY is the topological sum of a family $(X_y)_{y\in Y}$ of subsets of υY such that each $C(X_y)$ is isomorphic to $C(\{y\})$. But each $C(\{y\})$ is isomorphic to $\mathbf{R}$, so X_y is a singleton. Since Y is dense in υY, it follows that $Y=\upsilon Y$. Thus Y is realcompact, so $\mathfrak{m}$ is nonmeasurable by 2.1 (c).

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